ENVELOPING ACTIONS AND TAKAI DUALITY FOR PARTIAL ACTIONS

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ABSTRACT. We show that any continuous partial action α on a topological space X is the restriction of a suitable continuous global action α^{e}, that is essentially unique. We call this action α^{e} the enveloping action of α, and the space X^{e} where α^{e} acts is called the enveloping space of X. X^{e} is Hausdorff if and only if X is Hausdorff and the graph of α is closed.

In the case of C^*-algebras, we prove that any partial action has a unique enveloping action up to Morita equivalence, and that the corresponding reduced crossed products are Morita equivalent. The study of the enveloping action up to Morita equivalence reveals the form that Takai duality takes for partial actions.

By applying our constructions, we prove that any partial action of a connected group on a unital C^*-algebra must be a global action. We also prove that the reduced crossed product of the reduced cross sectional algebra of a Fell bundle by the dual coaction is liminal, postliminal, or nuclear, if and only if the unit fiber of the bundle is liminal, postliminal, or nuclear, respectively.

CONTENTS

1. INTRODUCTION

Partial actions on C^*-algebras were gradually introduced in [5], [18] and [9]. Since then, several classes of C^*-algebras have been described as crossed products by partial actions. This is the case of approximately finite, Bunce–Deddens and Cuntz–Krieger algebras, among others. ([7], [6], [11] and [12]). In addition, the description of a C^*-algebra as a crossed product by a partial action has proved to be useful to describe its structure and sometimes to compute its K-theory.

In the present paper we deal with enveloping actions of partial actions. That is, we discuss the problem of deciding whether or not a given partial action is the restriction of some global action, and the uniqueness of this global action.

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The exact statement of the problem depends on the category under consideration, as much as the definition of a partial action does. For instance, in the category of topological spaces and continuous maps, we say that an action β, acting on Y, is an enveloping action of the partial action α, acting on X, if X is an open subset of Y, $\alpha = \beta\big|_X$, and Y is the β–orbit of X. In the category of C^*-algebras and their homomorphisms, we say that (β, B) is an enveloping action of the partial action α, acting on the C^*-algebra A, if $A \lhd B$, $\alpha = \beta\big|_A$, and B is the closed linear β–orbit of A. In this paper we discuss enveloping actions in both categories. In the first one, the enveloping action always exists and is unique. In the second one it is unique when it exists, but in general this is not the case. For this reason we consider a weaker notion of enveloping action in the context of C^*-algebras, called Morita enveloping action. We show that any partial action on a C^*-algebra has a Morita enveloping action, which is unique up to Morita equivalence. Moreover, the corresponding reduced crossed products are strongly Morita equivalent. It turns out that Morita enveloping actions of partial actions are intimately related to Takai duality: if α is a partial action of the group G on a C^*-algebra A, δ is the dual coaction of G on $A \rtimes_{\alpha,r} G$ and $\hat{\delta}$ is the dual action of G on $A \rtimes_{\alpha,r} G \rtimes_{\delta,r} \hat{G}$, then $\hat{\delta}$ is the Morita enveloping action of α.

The structure of the paper is as follows.

In Section 2 we study partial actions in a topological context. In this case we show that for every partial action there exists a unique enveloping action, which is characterized by a universal property (2.5). We exhibit an example where the partial action acts on a Hausdorff space while its enveloping action acts on a non–Hausdorff space (2.9). This implies that in the category of C^*-algebras, the problem of existence of enveloping actions does not have a solution in general. Those partial actions whose enveloping actions act on a Hausdorff space are precisely those with closed graph (2.10).

Section 3 is devoted to consider the problem of enveloping actions in the category of C^*-algebras. In view of the results of Section 2, there is in general no enveloping action for a given partial action on a C^*-algebra. So the main matter of this section is the uniqueness of the enveloping action. It is shown in Theorem 3.8 that the enveloping action is unique when it exists. In the remainder of the section we study some relations between the C^*-algebras where a partial action and its enveloping action, respectively, act.

In Section 4 we discuss the relation between the reduced crossed products $A \rtimes_{\alpha,r} G$ and $B \rtimes_{\beta,r} G$, where (β, B) is the enveloping action of (α, A). In Theorem 4.18 we prove that they are Morita equivalent (In this paper Morita equivalence means strong Morita equivalence). As an application of this result, we show in 4.20 that any partial representation of a discrete amenable group may be dilated to a unitary representation of G, so in particular it is a positive definite map.

In order to overcome the negative result obtained in Section 4 about the existence of enveloping actions, we introduce in Section 5 the weaker notion of Morita enveloping action. This concept involves Morita equivalence of partial actions, which is defined and studied in this section. In particular, we show that the reduced crossed products of Morita equivalent partial actions are Morita equivalent (5.15). This allows us to deduce that the reduced crossed product by a partial action and the reduced crossed product by a corresponding Morita enveloping action are Morita equivalent (5.17).

In the sixth section we pave the way for proving the main result of the paper, namely the existence and uniqueness of the Morita enveloping action, which is achieved in Section 7 (7.3, 7.6). With this goal in mind, we consider two C^*-algebras, $\Bbbk(\mathcal{B})$ and $\Bbbk_r(\mathcal{B})$, that are completions of certain *-algebra of integral operators naturally associated to a Fell bundle. By using the uniqueness of the enveloping action, we prove that in fact these C^*-algebras are equal (6.16). In the last section we will see that, if the Fell bundle is associated to a partial action α on a C^*-algebra A, then the algebra $\Bbbk(\mathcal{B})$ also agrees with the double crossed product $A \rtimes_{\alpha,r} G \rtimes_{\delta,r} \hat{G}$, where δ is the dual coaction on $A \rtimes_{\alpha,r} G$ (9.1).

This paper corresponds to the second part of my doctoral thesis ([1]). It is a pleasure to express my gratitude to my advisor Ruy Exel for his guidance and several conversations about enveloping actions, which greatly enriched this work.

2. Enveloping actions: the topological case

In this section we consider the problem of enveloping actions in the category of topological spaces and continuous maps. In the first part we give the necessary definitions and some examples. In the second one we show that any partial action has a unique enveloping action, which is characterized by a universal property. This result implies, in particular, that if $\mathbf{v}$ is a vector field on a smooth manifold X, then there exist a smooth manifold Y, a vector field $\mathbf{w}$ on Y, and an inclusion $\iota : X \to Y$, such that $\iota(X)$ is open in Y and $\mathbf{v} = \mathbf{w}\iota$. We show that if (α, X) is a partial action, where X is a Hausdorff space, and if (β, Y) is its enveloping action, then Y is a Hausdorff space if and only if the graph of α is closed.

2.1. Partial actions: basic facts and examples.

Definition 2.1. *A partial action of a topological group G on a topological space X is a pair $\alpha = \big(\{X_s\}_{s\in G}, \{\alpha_s\}_{s\in G}\big)$ such that:*

1. *X_t is open in X, and $\alpha_t : X_{t^{-1}} \to X_t$ is a homeomorphism, $\forall t \in G$.*
2. *The set $\Gamma_\alpha = \big\{(t,x) \in G \times X : t \in G, x \in X_{t^{-1}}\big\}$ is open in $G \times X$, and the function (also called α) $\alpha : \Gamma_\alpha \to X$ given by $(t,x) \longmapsto \alpha_t(x)$ is continuous.*
3. *α is a partial action, that is, $X_e = X$, and α_{st} is an extension of $\alpha_s\alpha_t$, $\forall s,t \in G$.*

If $\alpha = \big(\{X_t\}_{t\in G}, \{\alpha_t\}_{t\in G}\big)$ and $\beta = \big(\{Y_t\}_{t\in G}, \{\beta_t\}_{t\in G}\big)$ are partial actions of G on X and Y, we say that a continuous function $\phi : X \to Y$ is a morphism $\phi : \alpha \to \beta$ if $\phi(X_t) \subseteq Y_t$, and the following diagram commutes, $\forall t \in G$:

$$\begin{tikzcd}
X_{t^{-1}} \arrow[r, "\phi"] \arrow[d, "\alpha_t"'] & Y_{t^{-1}} \arrow[d, "\beta_t"] \\
X_t \arrow[r, "\phi"'] & Y_t
\end{tikzcd}$$

If we forget the topological structures of G and X, we say that α is a set theoretic partial action; note that in this case condition 2. is superfluous, and condition 1. amounts to saying that each α_t is a bijection.

Condition 3. above is equivalent to the following set of conditions (see Lemma 1.2 of [24]):

1. $\alpha_e = id_X$ and $\alpha_{t^{-1}} = \alpha_t^{-1}$, $\forall t \in G$.
2. $\alpha_t(X_{t^{-1}} \cap X_s) = X_t \cap X_{ts}$, $\forall s,t \in G$.
3. $\alpha_s\alpha_t : X_{t^{-1}} \cap X_{t^{-1}s^{-1}} \to X_s \cap X_{st}$ is a bijection, and $\alpha_s\alpha_t(x) = \alpha_{st}(x)$, $\forall x \in X_{t^{-1}} \cap X_{t^{-1}s^{-1}}$ and $\forall s,t \in G$.

Example 2.2. Let $\beta : G \times Y \to Y$ be a continuous global action and let X be an open subset of Y. Consider $\alpha = \beta\big|_X$, the "restriction" of β to X, that is: $X_t = X \cap \beta_t(X)$, and $\alpha_t : X_{t^{-1}} \to X_t$ such that $\alpha_t(x) = \beta_t(x)$, $\forall t \in G$, $x \in X_{t^{-1}}$. It is easy to verify that α is a partial action on X. In fact, the main result of this section shows that any partial action arises in this way. Note that, in particular, β may be identified with the partial action $\beta\big|_Y$.

Example 2.3. The flow of a differentiable vector field is a partial action. More precisely, consider a smooth vector field $\mathbf{v} : X \to TX$ on a manifold X, and for $x \in X$ let γ_x be the corresponding integral curve through x (i.e.: $\gamma_x(0) = x$), defined on its maximal interval (a_x, b_x). Let us define, for $t \in \mathbb{R}$: $X_{-t} = \big\{x \in X : t \in (a_x, b_x)\big\}$, $\alpha_t : X_{-t} \to X_t$ such that $\alpha_t(x) = \gamma_x(t)$, and $\alpha = \big(\{X_t\}_{t\in\mathbb{R}}, \{\alpha_t\}_{t\in\mathbb{R}}\big)$. Then α is a partial action of $\mathbb{R}$ on X.

It is well known that the integral curves of a vector field on a compact manifold X are defined on all of $\mathbb{R}$. This is a particular case of the next result, which in turn may be generalized to a theorem about partial actions on C^*-algebras to be proved later in Section 8 (Corollary 8.7)

Proposition 2.4. *Let α be a partial action of G on a compact space X. Then there exists an open subgroup H of G such that α restricted to H is a global action. In particular, if G is connected, α is a global action.*

Proof. Let $A_x = \{t \in G : x \in X_{t^{-1}}\}$, and $A = \cap_{x\in X} A_x$. It is clear that $e \in A$ and $st \in A$ whenever s, $t \in A$; that is, A is a submonoid of G. For every $x \in X$ there exist open neighborhoods $U_x \subseteq X$ of x and $V_x \subseteq G$ of e such that $V_x \times U_x \subseteq \Gamma_\alpha$, and $V_x = V_x^{-1}$. Since X is compact, there exist $x_1, \dots, x_n \in X$ such that $X = \cup_{j=1}^n U_{x_j}$. Consider now the neighborhood $V = \cap_{j=1}^n V_{x_j}$. Since V is symmetric and $V \subseteq A$, we have that $H = \cup_{n=1}^\infty V^n$ is an open subgroup of G contained in A.

As for the last assertion, just recall that the unique open subgroup of a connected group is the group itself. ☐

2.2. **Existence and uniqueness of enveloping actions.**

Theorem 2.5. *Let α be a partial action of G on X. Then there exists a pair $(\iota, \alpha^{\mathsf{e}})$ such that α^{e} is a continuous action of G on a topological space X^{e}, and $\iota : \alpha \to \alpha^{\mathsf{e}}$ is a morphism, such that for any morphism $\psi : \alpha \to \beta$, where β is a continuous action of G, there exists a unique morphism $\psi^{\mathsf{e}} : \alpha^{\mathsf{e}} \to \beta$ making the following diagram commutative:*

$$\begin{tikzcd} \alpha \arrow[rr, "\iota"] \arrow[dr, "\psi"'] & \arrow[d, phantom, "\circlearrowleft"] & \alpha^{\mathsf{e}} \arrow[dl, dashed, "\psi^{\mathsf{e}}"] \\ & \beta & \end{tikzcd}$$

Moreover, the pair $(\iota, \alpha^{\mathsf{e}})$ is unique up to canonical isomorphisms, and:

1. *$\iota(X)$ is open in X^{e}.*
2. *$\iota : X \to \iota(X)$ is a homeomorphism.*
3. *X^{e} is the α^{e}–orbit of $\iota(X)$.*

Proof. Let us consider the action $\gamma : G \times (G \times X) \to G \times X$ such that $\gamma_s(t, x) = (st, x)$, $\forall s, t \in G$, $x \in X$. We endow $G \times X$ with the product topology, so γ is a continuous action. Moreover, γ is compatible with respect to the equivalence relation $\sim$ on $G\times X$ given by: $(r, x) \sim (s, y) \iff x \in X_{r^{-1}s}$ and $\alpha_{s^{-1}r}(x) = y$. Thus γ induces a continuous action α^{e} of G on the quotient topological space $X^{\mathsf{e}} = (G \times X)/\sim$. Let $q : G \times X \to X^{\mathsf{e}}$ be the quotient map, and define $\iota : X \to X^{\mathsf{e}}$ such that $\iota(x) = q(e, x)$. Since the inclusion $X \hookrightarrow G \times X$ given by $x \longmapsto (e, x)$ is continuous, we have that ι also is. Moreover, if $x \in X_{t^{-1}}$,

$$\iota\big(\alpha_t(x)\big) = q\big(e, \alpha_t(x)\big) = q(t, x) = q\big(\gamma_t(e, x)\big) = \alpha^{\mathsf{e}}_t\big(q(e, x)\big) = \alpha^{\mathsf{e}}_t\big(\iota(x)\big),$$

so ι is a morphism.

We must prove now that the pair $(\iota, \alpha^{\mathsf{e}})$ has the claimed universal property. Note first that if $\beta : G \times Y \to Y$ is a continuous action and $\psi : X \to Y$ is any continuous function, then the map $\psi' : G \times X \to Y$ such that $\psi'(t, x) = \beta_t\big(\psi(x)\big)$ is a morphism $\gamma \to \beta$. Moreover, if $\psi : \alpha \to \beta$ is also a morphism, then ψ' is compatible with $\sim$: if $(r, x) \sim (s, y)$ in $G \times X$, since $\alpha_{s^{-1}r}(x) = y$, we have:

$$\beta_{s^{-1}}\big(\psi'(r, x)\big) = \beta_{s^{-1}}\big(\beta_r(\psi(x))\big) = \beta_{s^{-1}r}\big(\psi(x)\big) = \psi\big(\alpha_{s^{-1}r}(x)\big) = \psi(y),$$

and therefore: $\psi'(r, x) = \beta_s\big(\psi(y)\big) = \psi'(s, y)$. Thus ψ' induces a continuous map $\psi^{\mathsf{e}} : X^{\mathsf{e}} \to Y$, such that $\psi^{\mathsf{e}}\big(q(t, x)\big) = \beta_t(\psi(x))$, $\forall t \in G$, $x \in X$. We have that $\psi^{\mathsf{e}}\iota(x) = \psi^{\mathsf{e}}\big(q(e, x)\big) = \psi(x)$, and it is also clear that $\psi^{\mathsf{e}} : \alpha^{\mathsf{e}} \to \beta$ is a morphism, uniquely determined by the relation $\psi^{\mathsf{e}}\iota = \psi$.

Since the pair $(\iota, \alpha^{\mathsf{e}})$ is characterized by a universal property, it is unique up to isomorphisms (In categorical terms, α^{e} is a universal from α to $\mathfrak{F}$, where $\mathfrak{F} : \mathcal{A} \to \mathcal{PA}$ is the forgetful functor from the category of actions to the category of partial actions; see [28] for details).

It remains to prove the last three assertions of the statement. The third of them is clear, because $q(t, x) = \alpha^{\mathsf{e}}_t\big(\iota(x)\big)$, $\forall t \in G$, $x \in X$. As for the first and second, note that ι is clearly injective, so it suffices

to show that it is an open map. Let $U \subseteq X$ be an open subset. We have to show that $q^{-1}\big(\iota(U)\big)$ is open in $G \times X$. But $q^{-1}\big(\iota(U)\big) = \{(t,x) : (t,x) \sim (e,y) \text{ for some } y \in U\} = \{(t,x) : \alpha_t(x) \in U\} = \alpha^{-1}(U)$, which is open in Γ_α because α is continuous, and hence open in $G \times X$ because Γ_α is open in $G \times X$. ☐

Definition 2.6. *Let α be a partial action of G on X. We say that the action α^{e} provided by Theorem 2.5 is an enveloping action of α. We will also say that X^{e} is the enveloping space of X, ψ^{e} is the enveloping morphism of ψ, etc.*

Remark 2.7. Assume that $h : X \to X$ is a homeomorphism, so we have an action of $\mathbb{Z}$ on X. We may think of this action as a partial action of $\mathbb{R}_d$ on X, where $\mathbb{R}_d$ denotes the real numbers with the discrete topology. Indeed, define $X_s = X$ if $s \in \mathbb{Z}$, $X_s = \emptyset$ if $s \notin \mathbb{Z}$, and $\alpha_s : X_{-s} \to X_s$ as $\alpha_s = h^s$ if $s \in \mathbb{Z}$, $\alpha_s = \emptyset$ otherwise. Note that α is not a partial action of $\mathbb{R}$ on X, because $\mathbb{Z} \times X$ is not open in $\mathbb{R} \times X$. However, we can imitate the construction of the enveloping action made in the proof of 2.5 above, using $\mathbb{R}$ instead of $\mathbb{R}_d$, to obtain a global continuous action $\beta : \mathbb{R} \times (\mathbb{R} \times X)/\sim \to (\mathbb{R} \times X)/\sim$, such that $\beta_n(x) = \alpha_n(x)$, $\forall n \in \mathbb{Z}$, $x \in X$. This action β is called the *suspension of* h, and its construction is well known in dynamical systems theory (see [29], page 45).

From now on we will suppose, as we can, that $X \subseteq X^{\mathsf{e}}$. Since X^{e} is the α^{e}–orbit of X, we see that X and X^{e} share the same local properties. However, their global properties may be very different, as shown in the next two examples.

Example 2.8. Consider the action $\beta : \mathbb{Z} \times S^1 \to S^1$ given by the rotation by an irrational angle θ: $\beta_k(z) = e^{2\pi i k\theta} z$, $\forall k \in \mathbb{Z}$, $z \in S^1$. Let U be a nonempty open arc of S^1, $U \neq S^1$, and consider the partial action given by the restriction α of β to U (see Example 2.2). Since the action β is minimal, it follows that β is the enveloping action of α. This example shows that, even when X and X^{e} are similar locally, their global properties may be deeply different. In this case, for instance, the first homotopy groups of U and S^1 are different.

Example 2.9. Consider the partial action α of $\mathbb{Z}_2$ on the unit interval $X = [0,1]$, given by $\alpha_1 = id_X$, $\alpha_{-1} = id_V$, where $V = (a,1]$, $a > 0$. Let $\alpha^{\mathsf{e}} : G \times X^{\mathsf{e}} \to X^{\mathsf{e}}$ be the enveloping action of α. Consider $J = J^- \cup J^+ \subseteq \mathbb{R}^2$ with the relative topology, where $J^\pm = \{\pm 1\} \times [0,1]$. It is not difficult to see that X^{e} is the topological quotient space obtained from J by identifying the points $(1,t)$ and $(-1,t)$, $\forall t \in (a,1]$. Therefore, X^{e} is not a Hausdorff space: $(1,a)$ and $(-1,a)$ do not have disjoint neighborhoods. Note also that α^{e}_{-1} permutes $(1,t)$ and $(-1,t)$ for $t \in [0,a]$, and is the identity in the rest of X^{e}.

The obstruction for the enveloping space to be Hausdorff is made clear in the next proposition.

Proposition 2.10. *Let α be a partial action of G on the Hausdorff space X. Let $\mathrm{Gr}(\alpha)$ be the graph of α, that is $\mathrm{Gr}(\alpha) = \{(t,x,y) \in G \times X \times X : x \in X_{t^{-1}}, \alpha_t(x) = y\}$. Then X^{e} is a Hausdorff space if and only if $\mathrm{Gr}(\alpha)$ is a closed subset of $G \times X \times X$.*

Proof. Let us suppose that X^{e} is a Hausdorff space, and let $\big(t_i, x_i, \alpha(t_i,x_i)\big) \to (t,x,y) \in G \times X \times X$. In particular, $\alpha(t_i,x_i) \to y \in X$. Since α^{e} is continuous, $\alpha^{\mathsf{e}}(t_i,x_i) \to \alpha^{\mathsf{e}}(t,x)$, and it must be $y = \alpha(t,x)$ because of the uniqueness of limits in Hausdorff spaces.

Conversely, assume that $\mathrm{Gr}(\alpha)$ is closed in $G \times X \times X$, and let $x^{\mathsf{e}}, y^{\mathsf{e}} \in X^{\mathsf{e}}$. We want to show that if there does not exist disjoint open sets in X^{e}, each of them containing x^{e} or y^{e}, then $x^{\mathsf{e}} = y^{\mathsf{e}}$. Since α^{e}_t is a homeomorphism of X^{e}, by 3. of 2.5 we may suppose that $x^{\mathsf{e}} = x \in X$. Let $y \in X$, $t \in G$ be such that $\alpha^{\mathsf{e}}_t(y) = y^{\mathsf{e}}$. If every neighborhood of x intersects every neighborhood of y^{e}, then for any pair (U,V) of neighborhoods in X of x and y respectively, there exists $x_{U,V} \in U \cap \alpha^{\mathsf{e}}_t(V)$, say $x_{U,V} = \alpha^{\mathsf{e}}_t(y_{U,V})$, with $y_{U,V} \in V$. Consider the net $\{(t, y_{U,V}, x_{U,V})\}_{U,V} \subseteq \mathrm{Gr}(\alpha)$: it converges to (t,y,x), so $\alpha_t(y) = x$, because $\mathrm{Gr}(\alpha)$ is closed. Hence $x = y^{\mathsf{e}}$, and X^{e} is Hausdorff. ☐

Remark 2.11. if G is a discrete group, then $\mathrm{Gr}(\alpha)$ is closed in $G \times X \times X$ if and only if $\mathrm{Gr}(\alpha_t)$ is closed in $X \times X$, $\forall t \in G$.

Remark 2.12. As already seen in 2.3, the flow of a smooth vector field on a manifold is a partial action, indeed a smooth partial action. The enveloping space inherits a natural manifold structure, although not always Hausdorff, by translating the structure of the original manifold through the enveloping action. It would be interesting to characterize those vector fields whose flows have closed graphs. For such a vector field, one obtains a Hausdorff manifold that contains the original one as an open submanifold, and a vector field whose restriction to this submanifold is the original vector field. Note, however, that the inclusion of the original manifold in its enveloping one could be a bit complicated.

It is possible to exhibit examples of flows with closed graphs and flows with non-closed graphs.

2.3. **On the dynamical properties of the enveloping action.** Before closing this section we would like to make some brief comments about the dynamical behavior of the enveloping action.

Many of the algebraic and even dynamical notions related to global actions may be easily extended to the context of partial actions. For instance, it is possible to make sense of expressions such as transitive partial actions or minimal partial actions. To give an example, we say that a partial action α on a topological space X is minimal when each α–orbit is dense in X, that is, when $X = \overline{\{\alpha_t(x) : t \in X_{t^{-1}}\}}$, $\forall x \in X$. It is not difficult to show that the dynamical properties of α and α^{e} are in general the same, although we will not do it in this work. For instance, it is not hard to see that α is minimal if and only if α^{e} is minimal.

3. Enveloping actions: the C^*–case

In this section we consider partial actions on C^*-algebras. We begin by recalling the definition of a partial action in this context, and then we introduce a notion of enveloping action that corresponds, in the case of commutative C^*-algebras, to the concept of enveloping action treated in Section 2. Next, we discuss the existence and uniqueness of enveloping actions, and we close the section by studying some properties of the enveloping C^*-algebras. Throughout the rest of this paper, G will denote a locally compact Hausdorff group.

The most general definition of a partial action is the one given in [9], where the reader is referred to for more information. We recall it in 3.2 below.

Definition 3.1. *Let E be a Banach space, X a topological space, and for each $x \in X$, let E_x be a Banach subspace of E. We say that $\{E_x\}_{x\in X}$ is a continuous family if for any open subset U of E, the set $\{x \in X : U \cap E_x \neq \emptyset\}$ is open in X.*

If $\mathcal{E} = \{(x, v) \in X \times E : v \in E_x\}$, with the product topology, and $\pi : \mathcal{E} \to X$ is the natural projection, then $\{E_x\}_{x\in X}$ is a continuous family if and only if π is open; in this case, $(\mathcal{E}, \pi)$ is a Banach bundle (see [9], where the notion of continuous family was introduced). For details about Banach bundles and Fell bundles, we refer the reader to [14]. Note that our notation differs sometimes from that of [14]. Also observe that Fell bundles are called C^*-algebraic bundles in that book.

Definition 3.2. *Let G be a locally compact group and $\alpha = (\{D_t\}_{t\in G}, \{\alpha_t\}_{t\in G})$ a set theoretic partial action of G on the C^*-algebra A, where each D_t is an ideal of A and each α_t is an isomorphism of C^*-algebras. Consider $\mathcal{B}^{-1} = \{(t, b) \in G \times B : b \in D_{t^{-1}}\} \subseteq G \times A$ with the product topology. We say that α is a partial action of G on A if $\{D_t\}_{t\in G}$ is a continuous family and the map (also called) $\alpha : \mathcal{B}^{-1} \to A$ such that $(t, b) \longmapsto \alpha_t(b)$ is continuous (note that, being $\{D_t\}_{t\in G}$ a continuous family, $\mathcal{B}^{-1}$ is a Banach bundle). If $\alpha' = (\{D'_t\}_{t\in G}, \{\alpha'_t\}_{t\in G})$ is a partial action of G on A', a morphism $\phi : \alpha \to \alpha'$ is a homomorphism $\phi : A \to A'$ such that $\phi(A_t) \subseteq A'_t$, $\forall t \in G$.*

In [9], Exel has shown that if $\alpha = (\{D_t\}_{t\in G}, \{\alpha_t\}_{t\in G})$ is a partial action of G on the C^*-algebra A, the Banach bundle $\mathcal{B}_\alpha = \{(t, x) : x \in D_t\} \subseteq G \times A$, with the relative topology, is a Fell bundle with

product and involution given by (here $x_t\delta_t := (t, x_t) \in \mathcal{B}_\alpha$): $(x_t\delta_t) * (x_s\delta_s) = \alpha_t\left(\alpha_t^{-1}(x_t)x_s\right)\delta_{ts}$ and $(x_t\delta_t)^* = \alpha_t^{-1}(x_t^*)\delta_{t^{-1}}$ respectively. $\mathcal{B}_\alpha$ is called the *semidirect product* of A and G. We will also say that $\mathcal{B}_\alpha$ is the Fell bundle *associated* with α. The cross sectional C^*-algebra $C^*(\mathcal{B}_\alpha)$ of $\mathcal{B}_\alpha$ is called *crossed product* of A by α, and is denoted by $A \rtimes_\alpha G$.

Example 3.3. If $\beta : G \times B \to B$ is a continuous action and $A \lhd B$, then the restriction $\beta\big|_A$ of β to A (see 2.2) is a partial action of G on A. In particular, β may be identified with the partial action $\beta\big|_B$.

Let us concentrate for a moment on the case where $A = C_0(X)$, for some locally compact Hausdorff space X. Suppose that $\{X_t\}_{t\in G}$ is a family of open subsets of X, so $\{D_t\}_{t\in G}$ is a family of ideals in A, where $D_t = C_0(X_t)$, $\forall t \in G$. If G is a locally compact Hausdorff space, one can show that $\{D_t\}_{t\in G}$ is a continuous family if and only if the set $\Gamma = \{(t,x) : x \in X_t\} \subseteq G\times X$ is open with the product topology. Suppose in addition that G is a group. To give an isomorphism $\alpha_t : D_{t^{-1}} \to D_t$ is equivalent to give a homeomorphism $\hat\alpha_t : X_{t^{-1}} \to X_t$. Now, it is possible to show that a given family of isomorphisms $\{\alpha_t : D_{t^{-1}} \to D_t\}_{t\in G}$ is a partial action on A if and only if the corresponding family of homeomorphisms $\{\hat\alpha_t : X_{t^{-1}} \to X_t\}_{t\in G}$ is a partial action on X ([1], [3]).

In the situation above, if the partial action $\hat\alpha = (\{X_t\}_{t\in G}, \{\hat\alpha_t\}_{t\in G})$ has an enveloping action $\hat\beta = \hat\alpha^{\mathsf{e}}$ acting on the enveloping space Y, then A is an ideal of $B = C_0(Y)$, and the action β induced by $\hat\beta$ on B satisfies: $\beta\big|_A = \alpha$. Moreover, the β–linear orbit $[\beta(A)] := \operatorname{span}\{\beta_t(a) : a \in A, t \in G\}$ of A is dense in B, by the Stone–Weierstrass theorem. These facts justify the following definition.

Definition 3.4. *Let $\alpha = (\{D_t\}_{t\in G}, \{\alpha_t\}_{t\in G})$ be a partial action of G on the C^*-algebra A, and let β be a continuous action of G on a C^*-algebra B that contains A. We say that (β, B) is an enveloping action of (α, A) (in the category of C^*-algebras), if the following three properties are fulfilled:*

1. *A is an ideal of B (two–sided and closed, of course).*
2. *$\alpha = \beta\big|_A$, that is $D_t = A \cap \beta_t(A)$, and $\alpha_t(x) = \beta_t(x)$, $\forall t \in G$ and $x \in D_{t^{-1}}$.*
3. *$B = \overline{[\beta(A)]}$, where $[\beta(A)] := \operatorname{span}\left\{\beta_t(x) : t \in G, x \in A\right\}$.*

We then say that B is an enveloping C^*-algebra of A.

Proposition 3.5. *Let $\alpha = (\{D_t\}_{t\in G}, \{\alpha_t\}_{t\in G})$ be a partial action of G on a commutative C^*-algebra A, and let $\hat\alpha$ the corresponding partial action of G on $\hat A$. Then the following assertions are equivalent:*

1. *$\mathrm{Gr}(\hat\alpha)$ is closed in $G \times \hat A \times \hat A$.*
2. *α has an enveloping action in the category of commutative C^*-algebras.*
3. *α has an enveloping action in the category of C^*-algebras.*

Proof. It is clear that 2. implies 3., and 1. and 2. are equivalent by Proposition 2.10, because of the categorical equivalence between locally compact Hausdorff spaces and commutative C^*-algebras. To see that 3. implies 2, let us suppose that (β, B) is an enveloping action of the partial action α on an abelian C^*-algebra A. Since A is a commutative ideal of B, A is contained in the center $Z(B)$ of B: if $a \in A$, $b \in B$, and $(e_i)_{i\in I} \subseteq A$ is an approximate unit of A:

$$ab = \lim_i (ab)e_i = \lim_i a(be_i) = \lim_i (be_i)a = \lim_i b(e_ia) = ba.$$

Since $Z(B)$ is invariant, it follows that $\beta_t(A) \subseteq Z(B)$, $\forall t \in G$, thus $\operatorname{span}\{\beta_t(a) : t \in G, a \in A\} \subseteq Z(B)$. But then $B = \overline{\operatorname{span}}\{\beta_t(a) : t \in G, a \in A\} \subseteq Z(B) \subseteq B$, and therefore $B = Z(B)$, so B is abelian. □

In [12] several algebras generated by isometries satisfying certain relations are studied, and they are shown to be crossed products by partial actions. At the topological level, all these partial actions have enveloping actions acting on Hausdorff spaces, and therefore they have also enveloping actions at the C^*-algebra level by 3.5.

3.1. **On the uniqueness of enveloping actions.** Proposition 3.5 and Example 2.9 show that not every partial action on a C^*-algebra has an enveloping action. We will prove that at least the enveloping action is unique when it does exist. Note that we already know this in the commutative case.

Lemma 3.6. *Let $\{J_\lambda\}_{\lambda\in\Lambda}$ be a family of ideals of a C^*-algebra A, and consider $\|\cdot\|_\Lambda : A \to \mathbb{R}$ such that $\|a\|_\Lambda = \sup_{\lambda\in\Lambda}\{\|ax\| : x \in J_\lambda, \|x\| \leq 1\}$. Then $\|\cdot\|_\Lambda$ is a C^*-seminorm on A, such that $\|\cdot\|_\Lambda \leq \|\cdot\|$, and $\|\cdot\|_\Lambda$ is a norm iff $\overline{\operatorname{span}}\{x \in J_\lambda : \lambda \in \Lambda\}$ is an essential ideal of A. In this case, $\|\cdot\|_\Lambda = \|\cdot\|$.*

Proof. Let $B = \prod_{\lambda\in\Lambda} A_\lambda$, where $A_\lambda = A$, $\forall\lambda \in \Lambda$, and consider $J=\{x = (x_\lambda) \in B : x_\lambda \in J_\lambda, \forall\lambda \in \Lambda\}$. Then B is a C^*-algebra with $\|b\| = \sup_{\lambda\in\Lambda}\|b_\lambda\|$, and J is an ideal of B. In particular, J may be considered as a right Hilbert B-module with the inner product: $\langle x, y\rangle = x^*y$, so there is a homomorphism $\eta : B \to \mathcal{L}(J)$ given by $\eta(b)x = bx$. On the other hand, we have an inclusion $\iota : A \hookrightarrow B$ given by $\iota(a)_\lambda = a$, $\forall\lambda \in \Lambda$. Thus, we get a homomorphism $\tilde{\eta} = \eta\iota$, and therefore $\|\tilde{\eta}(a)\| \leq \|a\|$, $\forall a \in A$. But $\|\tilde{\eta}(a)\| = \sup\{\|\eta(a)x\| : x \in J, \|x\| \leq 1\} = \|a\|_\Lambda$. Finally, it is clear that $\tilde{\eta}$ is injective if and only if $\overline{\operatorname{span}}\{x \in J_\lambda : \lambda \in \Lambda\}$ is an essential ideal of A. □

Lemma 3.7. *Let $\alpha = (\{D_t\}_{t\in G}, \{\alpha_t\}_{t\in G})$ be a partial action of G, and assume that (β, B) and (γ, C) are enveloping actions of α. Then, $\forall a, b \in A$, $t \in G$, we have: $\beta_t(a)b = \gamma_t(a)b$ (note that both of these products belong to D_t).*

Proof. Let (u_i) be an approximate unit of $D_{t^{-1}}$. Then $u_ia \in D_{t^{-1}}$, $\forall i$, and since α is both the restriction of β and γ to A, we have: $\beta_t(a)b = \lim\beta_t(u_ia)b = \lim\alpha_t(u_ia)b = \lim\gamma_t(u_ia)b = \gamma_t(a)b$. □

Theorem 3.8. *Let (α, A) be a partial action of G, and assume that (β, B) and (γ, C) are enveloping actions of α. Then there exists a unique isomorphism $\phi : B \to C$ such that $\phi\beta_t = \gamma_t\phi$, $\forall t \in G$, and $\phi\big|_A = id_A$.*

Proof. For $s \in G$, let $\|\cdot\|_s : B \to \mathbb{R}$ and $\|\cdot\|^s : C \to \mathbb{R}$ be given by $\|b\|_s := \sup\{\|bx\| : x \in \beta_s(A), \|x\| \leq 1\}$ and $\|c\|^s := \sup\{\|cy\| : y \in \gamma_s(A), \|y\| \leq 1\}$. By Lemma 3.6, $\|\cdot\|_s$ and $\|\cdot\|^s$ are C^*-seminorms, and $\|\cdot\|_B = \sup_s\|\cdot\|_s$, $\|\cdot\|_C = \sup_s\|\cdot\|^s$.

Let $t_1, \ldots, t_n \in G$ and $a_1, \ldots, a_n \in A$. We want to show that $\|\sum_i \beta_{t_i}(a_i)\|_B = \|\sum_i \gamma_{t_i}(a_i)\|_C$. For this, it is enough to prove that $\|\sum_i \beta_{t_i}(a_i)\|_s = \|\sum_i \gamma_{t_i}(a_i)\|^s, \forall s \in G$. Let $s \in G$ and $a \in A$. By Lemma 3.7 we have:

$$\|\sum_i \beta_{t_i}(a_i)\beta_s(a)\| = \|\beta_s\big(\sum_i \beta_{s^{-1}t_i}(a_i)a\big)\| = \|\gamma_s\big(\sum_i \gamma_{s^{-1}t_i}(a_i)a\big)\| = \|\sum_i \gamma_{t_i}(a_i)\gamma_s(a)\|.$$

It follows that $\|\sum_i \beta_{t_i}(a_i)\|_s = \|\sum_i \gamma_{t_i}(a_i)\|^s$, $\forall s \in G$, and hence that $\|\sum_i \beta_{t_i}(a_i)\|_B = \|\sum_i \gamma_{t_i}(a_i)\|_C$. Thus, $\phi : [\beta(A)] \to [\gamma(A)]$ such that $\phi\big(\sum_i \beta_{t_i}(a_i)\big) = \sum_i \gamma_{t_i}(a_i)$ is an isometry of a *-dense ideal of B onto a *-dense ideal of C, and therefore it extends uniquely to an isomorphism $\phi : B \to C$, which clearly satisfies $\gamma_t\phi = \phi\beta_t$, $\forall t \in G$, and $\phi\big|_A = id_A$. Moreover, it is clear that these conditions determine ϕ. □

3.2. **Some properties of the enveloping algebra.** To close this section we study some properties that are shared by a C^*-algebra and its enveloping algebra. In what follows it will be assumed that $(\alpha^{\mathrm{e}}, A^{\mathrm{e}})$ is an enveloping action of (α, A).

Proposition 3.9. *Let $\mathcal{C}$ be a class of C^*-algebras that is closed by ideals, isomorphisms, and such that any C^*-algebra B has a largest ideal $\mathcal{C}(B)$ that belongs to $\mathcal{C}$. Then $A \in \mathcal{C} \iff A^{\mathrm{e}} \in \mathcal{C}$, and $\mathcal{C}(A) = 0 \iff \mathcal{C}(A^{\mathrm{e}}) = 0$. ($\mathcal{C}$ may be, for instance, one of the following classes of C^*-algebras: nuclear, type I_0, liminal, postliminal, antiliminal). Then $A \in \mathcal{C} \iff A^{\mathrm{e}} \in \mathcal{C}$.*

Proof. Note that $\mathcal{C}(A^{\mathrm{e}})$ is α^{e}–invariant, and since $\mathcal{C}(A^{\mathrm{e}}) \cap A = \mathcal{C}(A)$, we have that $\mathcal{C}(A^{\mathrm{e}}) = \overline{\operatorname{span}}\{\alpha^{\mathrm{e}}_t\big(\mathcal{C}(A) : t \in G\big)\}$. From this, the result follows immediately. □

Proposition 3.10. *Any separable C^*-algebra has a largest ideal that is approximately finite. If G is a separable group, then A is approximately finite iff A^{e} is approximately finite.*

Proof. Let $AF(B)$ be the set of AF–ideals of a C^*-algebra B. Since $0 \in AF(B)$, we have that $AF(B) \neq \emptyset$. Let $I, J \in AF(B)$, and consider the following exact sequence of C^*-algebras:

$$0 \longrightarrow I \xrightarrow{\ \iota\ } I + J \xrightarrow{\ \pi\ } J/(I \cap J) \longrightarrow 0,$$

where ι is the inclusion and π is the quotient map. Since the class of AF–C^*-algebras is closed by ideals, quotients and extensions, it follows that $I + J \in AF(B)$. Suppose in addition that B is a separable C^*-algebra, and let $D = \{d_n\}_{n\geq 1}$ be a countable and dense subset of $\operatorname{span}\{x \in J : J \in AF(B)\}$. Since $J_1 + \ldots + J_k \in AF(B)$, whenever $J_1, \ldots, J_k \in AF(B)$, there exists an increasing sequence $\{J_n\}_{n\geq 1} \subseteq AF(B)$ such that $d_n \in J_n$, $\forall n \geq 1$. It follows that $J = \overline{\cup_{n\geq 1} J_n}$ is an AF–ideal that contains any ideal of $AF(B)$, and this proves our first assertion. As for the second one, note that, since G is separable, then A is separable if and only if so is A^{e}. Now, by the first part, the result is proved as in 3.9. □

4. Enveloping actions and crossed products

The main result of this section, Theorem 4.18, is the following: if a partial action α of G on a C^*-algebra A has an enveloping action $(\alpha^{\mathrm{e}}, A^{\mathrm{e}})$, then $A \rtimes_{\alpha,r} G$ and $A^{\mathrm{e}} \rtimes_{\alpha^{\mathrm{e}},r} G$ are Morita equivalent. This theorem will be obtained as a consequence of a more general result on Fell bundles.

We begin by recalling the definition of the regular representation and the reduced cross sectional algebra of a Fell bundle. Then we prove 4.9, a result that will be of great importance later to prove 4.18. Finally we apply Theorem 4.18 to show that any partial representation of an amenable discrete group G is the compression of some unitary representation of G.

4.1. **Preliminaries on Fell bundles and crossed products by partial actions.** In [13], the authors defined the reduced cross sectional algebra of a Fell bundle, generalizing the definition given by Exel in [10] for bundles over discrete groups. The definition is the following: $C^*_r(\mathcal{B})$ is the closure of $\Lambda\big(L^1(\mathcal{B})\big)$ in $\mathcal{L}\big(L^2(\mathcal{B})\big)$, where $\Lambda_f\xi = f * \xi$, $\forall f \in C_c(\mathcal{B}) \subseteq L^1(\mathcal{B})$, $\xi \in C_c(\mathcal{B}) \subseteq L^2(\mathcal{B})$. When G is a non–discrete group, it is not immediate that Λ, defined by means of convolution of continuous functions, extends to a representation of $L^1(\mathcal{B})$. The proof of this fact given below seems to us more direct than the one in [13]. Next, we show that if $\mathcal{A}$ is a sub–Fell bundle (4.8) of $\mathcal{B}$, then $C^*_r(\mathcal{A})$ may be considered to be a sub-C^*-algebra of $C^*_r(\mathcal{B})$.

For the general theory of Fell bundles, also called C^*–algebraic bundles, and their representations, the reader is referred to [14]. Let us suppose that $\mathcal{B} = (B_t)_{t\in G}$ is a Fell bundle over G. If $K \subseteq G$ is a compact subset, then $C_K(\mathcal{B})$ denotes the Banach space of continuous sections of $\mathcal{B}$, with the supremum norm. $C_c(\mathcal{B})$ denotes the *–algebra of continuous sections of $\mathcal{B}$ that have compact support, with the locally convex inductive limit topology defined by the natural inclusions $C_K(\mathcal{B}) \overset{\iota_K}{\hookrightarrow} C_c(\mathcal{B})$. If B_e is the fiber of $\mathcal{B}$ over the unit e of G, then $L^2(\mathcal{B})$ is the right Hilbert B_e–module obtained by completing $C_c(\mathcal{B})$ with respect to the B_e–inner product: $\langle \xi, \eta\rangle = \int_G \xi(s)^*\eta(s)ds$. If $b_t \in B_t$, let $(\Lambda_{b_t}\xi)\big|_s = b_t\xi(t^{-1}s)$, $\forall \xi \in C_c(\mathcal{B})$. We have that $\Lambda_{b_t}\xi \in C_c(\mathcal{B})$, and $\operatorname{supp}(\Lambda_{b_t}\xi) \subseteq t\operatorname{supp}(\xi)$. In addition: $\langle \Lambda_{b_t}\xi, \Lambda_{b_t}\xi\rangle = \int_G \xi(t^{-1}s)^* b_t^* b_t \xi(t^{-1}s)ds \leq \int_G \xi(t^{-1}s)^*\|b_t^*b_t\|\xi(t^{-1}s)ds = \|b_t\|^2\langle\xi,\xi\rangle$, and hence Λ_{b_t} may be extended to $L^2(\mathcal{B})$. In fact, it is easy to see that Λ_{b_t} is adjointable, and $\Lambda^*_{b_t} = \Lambda_{b_t^*}$.

Let us define $\Lambda : \mathcal{B} \to \mathcal{L}\big(L^2(\mathcal{B})\big)$ by $b \longmapsto \Lambda_b$. It is immediate that $\Lambda\big|_{B_t}$ is a bounded linear map, $\forall t \in G$, and that $\Lambda_b\Lambda_c = \Lambda_{bc}$, $\forall b, c \in \mathcal{B}$. We will see that Λ is also continuous (that is: Λ satisfies 4.2).

Lemma 4.1. *If $\xi \in C_c(\mathcal{B})$ and $b_t \in \mathcal{B}$, then for any $\epsilon > 0$ there exists an open $U \subseteq \mathcal{B}$, with $b_t \in U$, such that if $b \in U$, then $\|\Lambda_b\xi - \Lambda_{b_t}\xi\|_\infty < \epsilon$.*

Proof. Suppose that there exists $\epsilon > 0$ such that for any open neighborhood U of b_t there exist $b_{r_U} \in U$ and $s_U \in G$ such that $\|(\Lambda_{b_{r_U}}\xi)(s_U) - (\Lambda_{b_t}\xi)(s_U)\| \geq \epsilon$, that is, $\|b_{r_U}\xi(r_U^{-1}s_U) - b_t\xi(t^{-1}s_U)\| \geq \epsilon$. Note

that $\operatorname{supp}(\Lambda_{b_{r_U}}\xi)\subseteq r_U\operatorname{supp}(\xi)$, $\operatorname{supp}(\Lambda_{b_t}\xi)\subseteq t\operatorname{supp}(\xi)$. Since $b_{r_U}\to b_t$, then $r_U\to t$. Thus there exist a compact set $K\subseteq G$ and a neighborhood U_0 of b_t such that $\operatorname{supp}(\Lambda_{b_{r_U}}\xi)\subseteq K$, $\forall U\subseteq U_0$. Then the net $(s_U)_{U\subseteq U_0}\subseteq K$, and hence it must have a subnet that is convergent to some $s_0\in K$. We may assume without loss of generality that the net itself converges to s_0. But this is a contradiction, because:

$$0=\|b_t\xi(t^{-1}s_0)-b_t\xi(t^{-1}s_0)\|=\lim_U\|b_{r_U}\xi(r_U^{-1}s_U)-b_t\xi(t^{-1}s_U)\|\geq\epsilon.$$

The contradiction implies that the Lemma is true. □

Recall that $\mathfrak{L}^2(\mathcal{B})$ is the completion of $C_c(\mathcal{B})$ with respect to the norm $\|\xi\|_2=\left(\int_G\|\xi(s)\|^2ds\right)^{1/2}$. So if $\xi_n\to\xi$ in $\mathfrak{L}^2(\mathcal{B})$, with $\xi_n,\xi\in C_c(\mathcal{B})$, we have that $\xi_n\to\xi$ in $L^2(\mathcal{B})$, because $\|\xi\|\leq\|\xi\|_2$. In fact, $\|\xi\|=\|\int_G\xi(s)^*\xi(s)ds\|^{1/2}$, and since $\int_G\xi(s)^*\xi(s)ds$ is a positive element of B_e, there is a state φ of B_e such that $\|\int_G\xi(s)^*\xi(s)ds\|=\varphi(\int_G\xi(s)^*\xi(s)ds)$. Then:

$$\left\|\int_G\xi(s)^*\xi(s)ds\right\|=\varphi\left(\int_G\xi(s)^*\xi(s)ds\right)=\int_G\varphi\left(\xi(s)^*\xi(s)\right)ds\leq\int_G\|\xi(s)^*\xi(s)\|ds=\|\xi\|_2^2$$

On the other hand, it is clear that if $b_r\to b_t$, then $\Lambda_{b_r}\xi\to\Lambda_{b_t}\xi$ in $\|\cdot\|$, because $\|\xi\|_2\leq m\big(\operatorname{supp}(\xi)\big)\|\xi\|_\infty$ (here m is the left Haar measure on G), and hence, by Lemma 4.1, $\Lambda_{b_r}\xi\to\Lambda_{b_t}\xi$ in $\|\cdot\|_\infty$, so $\Lambda_{b_r}\xi\to\Lambda_{b_t}\xi$ in $\|\cdot\|_2$ and $\|\cdot\|$.

Proposition 4.2. *Let $\xi\in L^2(\mathcal{B})$. Then the map $\mathcal{B}\to L^2(\mathcal{B})$, given by $b\longmapsto\Lambda_b\xi$, is continuous.*

Proof. Let us fix $b\in\mathcal{B}$, and let $b_j\to b$ in $\mathcal{B}$. Given $\epsilon>0$, let $\xi_\epsilon\in C_c(\mathcal{B})$ such that $\|\xi-\xi_\epsilon\|<\epsilon$, and let j_0 such that $\|b_j\|$, $\|b\|\leq c$, $\forall j\geq j_0$ and some constant c. Then, if $j\geq j_0$:

$$\|\Lambda_{b_j}\xi-\Lambda_b\xi\|\leq\|b_j\|\,\|\xi-\xi_\epsilon\|+\|\Lambda_{b_j}\xi_\epsilon-\Lambda_b\xi_\epsilon\|+\|b\|\,\|\xi_\epsilon-\xi\|<2c\varepsilon+\|\Lambda_{b_j}\xi_\epsilon-\Lambda_b\xi_\epsilon\|.$$

It follows that $\limsup_i\|\Lambda_{b_j}\xi-\Lambda_b\xi\|\leq 2c\epsilon$, $\forall\epsilon>0$, and therefore $\Lambda_{b_j}\xi\to\Lambda_b\xi$ in $\|\cdot\|$. □

Definition 4.3. *(cf. [13]) The representation $\Lambda:\mathcal{B}\to\mathcal{L}\big(L^2(\mathcal{B})\big)$ defined above is called the regular representation of the Fell bundle $\mathcal{B}$. That is, Λ_{b_s} is the unique continuous extension to all of $L^2(\mathcal{B})$ of the map $C_c(\mathcal{B})\to C_c(\mathcal{B})$ such that, if $\xi\in C_c(\mathcal{B})$, $t\in G$, then $\Lambda_{b_s}(\xi)\big|_t=b_s\xi(s^{-1}t)$.*

Theorem 4.4. *(cf. [13]) There exists a unique non–degenerate representation $\Lambda:L^1(\mathcal{B})\to\mathcal{L}\big(L^2(\mathcal{B})\big)$, given by $f\longmapsto\Lambda_f$, where $\Lambda_f(\xi)=f*\xi$, $\forall f\in C_c(\mathcal{B})\subseteq L^1(\mathcal{B})$, $\xi\in C_c(\mathcal{B})\subseteq L^2(\mathcal{B})$.*

Proof. Proposition 4.2 tells us that $\Lambda:\mathcal{B}\to\mathcal{L}\big(L^2(\mathcal{B})\big)$ is a Fréchet representation, in the sense of VIII-8.2 of [14]. So we may apply VIII-11.3 of [14], and conclude that Λ is integrable. That is, there exists a representation $\Lambda:C_c(\mathcal{B})\to B\big(L^2(\mathcal{B})\big)$ such that $\varphi(\Lambda_f)=\int_G\varphi(\Lambda_{f(s)})ds,\ \ \forall f\in C_c(\mathcal{B}),\ \varphi\in B\big(L^2(\mathcal{B})\big)'$. Moreover, Λ is unique. We set $\Lambda_f=\int_G\Lambda_{f(s)}ds$.

We want to see that $\forall f\in C_c(\mathcal{B})$, it is $\Lambda_f\in\mathcal{L}\big(L^2(\mathcal{B})\big)$. Now, for $\xi,\eta\in L^2(\mathcal{B})$, we have that $\langle\xi,\Lambda_f(\eta)\rangle=\int_G\langle\xi,\Lambda_{f(s)}(\eta)\rangle ds$. In particular, since $f^*(s)=\Delta(s^{-1})f(s^{-1})^*$,

$$\langle\xi,\Lambda_{f^*}(\eta)\rangle=\int_G\langle\xi,\Lambda_{\Delta(s^{-1})f(s^{-1})^*}(\eta)\rangle ds=\int_G\Delta(s^{-1})\left(\Delta(s)\langle\Lambda_{f(s)}(\xi),\eta\rangle\right)ds=\langle\Lambda_f(\xi),\eta\rangle.$$

Thus $\Lambda_f^*=\Lambda_{f^*}$, and therefore $\Lambda_f\in\mathcal{L}\big(L^2(\mathcal{B})\big)$. Moreover, the representation $\Lambda:C_c(\mathcal{B})\to\mathcal{L}\big(L^2(\mathcal{B})\big)$ is continuous in the norm $\|\cdot\|_1$: $\|\langle\xi,\Lambda_f(\eta)\rangle\|\leq\int_G\|\xi\|\,\|\Lambda_{f(t)}\eta\|dt\leq\int_G\|f(t)\|\,\|\xi\|\,\|\eta\|dt=\|f\|_1\,\|\xi\|\,\|\eta\|$. It follows that $\|\Lambda_f\|\leq\|f\|_1$, and hence we may extend Λ by continuity to a representation of $L^1(\mathcal{B})$. This representation is non–degenerate, because $C_c(\mathcal{B})*C_c(\mathcal{B})$ is dense in $C_c(\mathcal{B})$ in the inductive limit topology, and therefore also in $L^2(\mathcal{B})$. □

Definition 4.5. *(cf. [13]) The representation $\Lambda : L^1(\mathcal{B}) \to \mathcal{L}\big(L^2(\mathcal{B})\big)$ defined in Theorem 4.4 is called the regular representation of $L^1(\mathcal{B})$, and $C^*_r(\mathcal{B}) := \overline{\Lambda\big(L^1(\mathcal{B})\big)} \subseteq \mathcal{L}\big(L^2(\mathcal{B})\big)$ is called the reduced C^*-algebra of $\mathcal{B}$. If $\mathcal{B}_\alpha$ is the Fell bundle associated with a partial action α of G on a C^*-algebra A, then its reduced C^*-algebra is called the reduced crossed product of A by α, and it is denoted by $A \rtimes_{\alpha,r} G$.*

Remark 4.6. It is shown in [13] that if α is a global action of G on A, then the usual reduced crossed product agrees with the one defined in 4.5 (see also [10], 3.8). Moreover, the authors show that if $\pi : \mathcal{B} \to B(H)$ is a non–degenerate representation of $\mathcal{B}$ and if π_λ is the representation $\pi_\lambda : \mathcal{B} \to B\left(L^2(G,H)\right)$, given by $\pi_\lambda(b_t) = \lambda_t \otimes \pi(b_t)$, where λ is the left regular representation of G on $L^2(G)$, then the integrated representation of π_λ defines a representation of $C^*(\mathcal{B})$, that factors through $C^*_r(\mathcal{B})$. In addition, if $\pi\big|_{B_e}$ is faithful, we have that $C^*_r(\mathcal{B}) \cong \pi_\lambda\big(C^*(\mathcal{B})\big)$ (2.15 of [13]). Note that if π is a degenerate representation, the result is also true, because in this case $\pi = \rho \oplus 0$, the direct sum of the non degenerate part of π with a zero representation, and therefore $\pi_\lambda = \rho_\lambda \oplus 0$.

Let us recall the definition of amenable Fell bundle ([10], [13]):

Definition 4.7. *The regular representation Λ induces a representation of $C^*(\mathcal{B})$, also called regular and denoted by Λ. When $\Lambda : C^*(\mathcal{B}) \to C^*_r(\mathcal{B})$ is an isomorphism, we say that $\mathcal{B}$ is amenable. If $\mathcal{B}_\alpha$ is the Fell bundle associated with the partial action α, we say that α is amenable when $\mathcal{B}_\alpha$ is amenable.*

Definition 4.8. *If $\mathcal{B} = (B_t)_{t\in G}$ is a Banach bundle over the Hausdorff space G (or a Fell bundle over G), we say that $\mathcal{A} \subseteq \mathcal{B}$ is a sub–Banach bundle of $\mathcal{B}$ (respectively: a sub–Fell bundle of $\mathcal{B}$) if it is a Banach (respectively: Fell) bundle over G with the structure inherited from $\mathcal{B}$.*

Proposition 4.9. *If $\mathcal{A}$ is a sub–Fell bundle of $\mathcal{B}$, then $C^*_r(\mathcal{A}) \subseteq C^*_r(\mathcal{B})$. More precisely: the closure of $C_c(\mathcal{A})$ in $C^*_r(\mathcal{B})$ is naturally isomorphic to $C^*_r(\mathcal{A})$.*

Proof. Let $\pi : \mathcal{B} \to B(H)$ be a representation of $\mathcal{B}$ on the Hilbert space H, such that $\pi\big|_{B_e}$ is faithful. Then $\rho := \pi\big|_{\mathcal{A}} : \mathcal{A} \to B(H)$ is a representation of $\mathcal{A}$, such that $\rho\big|_{A_e}$ is faithful. Let $\pi_\lambda : \mathcal{B} \to B\big(L^2(G) \bigotimes H\big)$ such that $\pi_\lambda(b_t) = \lambda_t \otimes \pi(b_t)$, where $\lambda : G \to B\big(L^2(G)\big)$ is the left regular representation of G; that is: $\forall \xi \in L^2(G)$, $\lambda_t(\xi)\big|_s = \xi(t^{-1}s)$. Similarly, define $\rho_\lambda : \mathcal{A} \to B\big(L^2(G) \bigotimes H\big)$. It is clear that $\rho_\lambda = \pi_\lambda\big|_{\mathcal{A}}$. Integrating π_λ and ρ_λ, we obtain representations of $C^*(\mathcal{B})$ and $C^*(\mathcal{A})$, which we also call π_λ and ρ_λ respectively, and it is clear again that $\rho_\lambda\big|_{L^1(\mathcal{A})}$ agrees with $\pi_\lambda\big|_{L^1(\mathcal{A})}$. Now, by 2.15 of [13] (see also the end of Remark 4.6), we have isomorphisms $\tilde{\pi}_\lambda : C^*_r(\mathcal{B}) \to \overline{\pi_\lambda\big(C^*(\mathcal{B})\big)}$, and $\tilde{\rho}_\lambda : C^*_r(\mathcal{A}) \to \overline{\rho_\lambda\big(L^1(\mathcal{A})\big)} \cong C^*_r(\mathcal{A})$. Therefore, $\tilde{\pi}_\lambda^{-1}\tilde{\rho}_\lambda : C^*_r(\mathcal{A}) \to \overline{C_c(\mathcal{A})} \subseteq C^*_r(\mathcal{B})$ is an isomorphism. Thus, $C^*_r(\mathcal{A})$ is naturally identified with $\overline{C_c(\mathcal{A})}$ in $C^*_r(\mathcal{B})$. $\square$

Remark 4.10. When G is a discrete group, there is a shorter proof of 4.9, because in this case the reduced cross sectional algebra is characterized in terms of conditional expectations ([10]).

Definition 4.11. *Let $\mathcal{A} = (A_t)_{t\in G}$ and $\mathcal{B} = (B_t)_{t\in G}$ be Banach bundles. A homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ is a continuous map such that $\phi(A_t) \subseteq B_t$, and $\phi\big|_{A_t} : A_t \to B_t$ is linear and bounded, $\forall t \in G$. If $\mathcal{A}$ and $\mathcal{B}$ are Fell bundles, we also require that: $\phi(xy) = \phi(x)\phi(y)$, $\phi(x^*) = \phi(x)^*$, $\forall x, y \in \mathcal{A}$.*

Remark 4.12. Let $\phi : \mathcal{A} \to \mathcal{B}$ be a homomorphism. If $f \in L^1(\mathcal{A})$, we have that $\phi^1(f) : G \to \mathcal{B}$ such that $\phi^1(f)(t) = \phi\big(f(t)\big)$ is in $L^1(\mathcal{B})$, and $\|\phi^1(f)\|_1 \leq \|f\|_1$. On the other hand, it is clear that ϕ^1 is a homomorphism of Banach *–algebras, and therefore it extends to a homomorphism $C^*(\phi) : C^*(\mathcal{A}) \to C^*(\mathcal{B})$. This way we obtain a functor from the category of Fell bundles over G to the category of C^*-algebras. A morphism $\phi : \alpha \to \beta$ between partial actions induces a homomorphism $\phi : \mathcal{B}_\alpha \to \mathcal{B}_\beta$ between the corresponding Fell bundles, given by $\phi\big((t, a_t)\big) = \big(t, \phi(a_t)\big)$. Thus we have a functor from the category of partial actions to the category of C^*-algebras.

4.2. **Morita equivalence between the reduced crossed products.**

Definition 4.13. *If $\mathcal{B}=(B_t)_{t\in G}$ is a Fell bundle, we say that a sub–Banach bundle $\mathcal{A}$ of $\mathcal{B}$ (4.8) is a right ideal of $\mathcal{B}$ if $\mathcal{A}\mathcal{B}\subseteq\mathcal{A}$, and that it is a left ideal if $\mathcal{B}\mathcal{A}\subseteq\mathcal{A}$; $\mathcal{A}$ is said to be an ideal of $\mathcal{B}$ if it is both a right and a left ideal of $\mathcal{B}$.*

Consider a Fell bundle $\mathcal{B}=(B_t)_{t\in G}$. If R is a right ideal of B_e and we define $R_t:=\overline{\operatorname{span}}RB_t$, $\forall t\in G$, we obtain a right ideal $\mathcal{R}=(R_t)_{t\in G}$ of $\mathcal{B}$. In a similar way we may use left ideals of B_e to define left ideals of $\mathcal{B}$. If $I\lhd B_e$, then $\mathcal{I}=(I_t)_{t\in G}$, where $I_t=IB_t$, is a right ideal of $\mathcal{B}$, but in general not an ideal. For this it is necessary and sufficient that I is a $\mathcal{B}$–*invariant* ideal of B_e, that is, $IB_t=B_tI$, $\forall t\in G$. Conversely, if $\mathcal{I}=(I_t)_{t\in G}$ is a given ideal of $\mathcal{B}$, and $I=I_e$, then I is a $\mathcal{B}$–invariant ideal of B_e. Moreover, these correspondences establish an inclusion–preserving bijection between $\mathcal{B}$–invariant ideals of B_e and ideals of $\mathcal{B}$.

Let $\mathcal{V}$ be a local base of precompact and symmetric neighborhoods of the unit $e\in G$, directed by the relation: $V\geq V'\iff V\subseteq V'$. Then there exists an approximate unit $(f_V)_{V\in\mathcal{V}}$ of $L^1(G)$ contained in $C_c(G)$, and such that $\operatorname{supp}(f_V)\subseteq V$, $f_V\geq 0$, and $\int_G f_V(s)ds=1$, $\forall V\in\mathcal{V}$.

Lemma 4.14. *Let $\mathcal{B}=(B_t)_{t\in G}$ be a Fell bundle, $b:G\times G\to\mathcal{B}$ a compactly supported continuous function such that $b(r,s)\in B_s$, $\forall(r,s)\in G\times G$. Let $(f_V)_{V\in\mathcal{V}}$ be an approximate unit of $L^1(G)$ as described just above, and define, for each pair $(V,r)\in\mathcal{V}\times G$, the function $b_{V,r}:G\to\mathcal{B}$, such that $b_{V,r}(t)=\int_G f_V(r^{-1}s)b(s,t)ds$. Then:*

1. *$b_{V,r}\in C_c(\mathcal{B})$, $\forall V\in\mathcal{V}$, $r\in G$.*
2. *$\lim_V b_{V,r}=b_r$ in the inductive limit topology, where $b_r:G\to\mathcal{B}$ is given by $b_r(s)=b(r,s)$.*

Proof. The function $\mu:G\times G\times G\to\mathcal{B}$ such that $(r,t,s)\longmapsto f_V(r^{-1}s)b(s,t)$ is continuous and $\mu(r,t,s)\in B_s$, $\forall r,s,t\in G$. So by [14], II-15.19, the function $G\times G\to\mathcal{B}$ given by $(r,t)\longmapsto\int_G f_V(r^{-1}s)b(s,t)ds$ is continuous. In particular, $b_{V,r}$ is continuous, and thus 1. is proved.

As for 2., let K_1, $K_2\subseteq G$ be compact sets such that $\operatorname{supp}(b)\subseteq K_1\times K_2$. Since the function b is continuous, each $b_r:G\to\mathcal{B}$ is a continuous section with $\operatorname{supp}(b_r)\subseteq K_2$, so we have a function $G\to C_c(\mathcal{B})$ defined by $r\to b_r$, that is supported in K_1, and that is continuous. In fact, let $r_0\in G$. Since the function $G\times G\to\mathbb{R}$ that maps (r,s) into $\|b(r_0,s)-b(r,s)\|$ is continuous and equal to zero in every (r_0,s), for $s\in K_2$, there exist open neighborhoods U_s of r_0, V_s of s, such that $\|b(r_0,s')-b(r,s')\|<\epsilon$, $\forall r\in U_s$, $s'\in V_s$. Since the V_s cover the compact set K_2, there exists a finite subcovering $V_{s_1},\dots,V_{s_n}$ of K_2. Let $U=\bigcap_{i=1}^n U_{s_i}$ and pick $r\in U$, $s\in G$. Then either $s\notin K_2$, and then $b_r(s)=0$ $\forall r\in G$, or $s\in K_2$, and therefore $s\in V_{s_i}$ for some i. In this case, $(r,s)\in U_{s_i}\times V_{s_i}$, and hence $\|b_{r_0}(s)-b_r(s)\|<\epsilon$, so $\|b_{r_0}-b_r\|_\infty<\epsilon$. It follows that $r\longmapsto b_r$ is continuous, and hence uniformly continuous, because it has compact support. Thus, for any $\epsilon>0$, there exists $V_0\in\mathcal{V}$ such that if $r^{-1}s\in V_0$, then $\|b_r-b_s\|_\infty<\epsilon$. So we have, for any $V\in\mathcal{V}$ such that $V\geq V_0$, and for all t:

$$\|(b_r-b_{V,r})(t)\|=\|\int_G f_V(r^{-1}s)\big(b(r,t)-b(s,t)\big)ds\|\leq\int_V f_V(r^{-1}s)\|b(r,t)-b(s,t)\|ds<\epsilon$$

Therefore, since $\operatorname{supp}(b_r)$, $\operatorname{supp}(b_{V,r})\subseteq K_2$ and $\|b_r-b_{V,r}\|_\infty<\epsilon$, $\forall V\geq V_0$, it follows that $\lim_V b_{V,r}=b_r$ in the inductive limit topology, $\forall r\in G$. □

Theorem 4.15. *Let $\mathcal{B}=(B_t)_{t\in G}$ be a Fell bundle, $\mathcal{E}=(E_t)_{t\in G}$ a right ideal of $\mathcal{B}$ (4.13), and $\mathcal{A}=(A_t)_{t\in G}$ a sub–Fell bundle (4.8)of $\mathcal{B}$ contained in $\mathcal{E}$. If $\mathcal{A}\mathcal{E}\subseteq\mathcal{E}$ and $\mathcal{E}\mathcal{E}^*\subseteq\mathcal{A}$, we have:*

1. *$L^1(\mathcal{A})*L^1(\mathcal{E})\subseteq L^1(\mathcal{E})$*
2. *$L^1(\mathcal{E})*L^1(\mathcal{B})\subseteq L^1(\mathcal{E})$.*
3. *$L^1(\mathcal{E})*L^1(\mathcal{E})^*=L^1(\mathcal{A})$*
4. *If $\operatorname{span}(B_t\bigcap\mathcal{E}^*\mathcal{E})$ is dense in B_t, $\forall t\in G$, then $\overline{\operatorname{span}}L^1(\mathcal{E})^**L^1(\mathcal{E})=L^1(\mathcal{B})$.*

Proof. It is straightforward to check that $C_c(\mathcal{A})*C_c(\mathcal{E})\subseteq C_c(\mathcal{E})$ and $C_c(\mathcal{E})*C_c(\mathcal{B})\subseteq C_c(\mathcal{E})$, and from these facts the two first inclusions follow easily. Let us prove the third assertion. Since $\mathcal{A}\subseteq\mathcal{E}\subseteq\mathcal{B}$, we have isometric inclusions $L^1(\mathcal{A})\subseteq L^1(\mathcal{E})\subseteq L^1(\mathcal{B})$. $L^1(\mathcal{A})$ is a sub-*-Banach algebra with approximate unit of $L^1(\mathcal{B})$, and it is contained in the right ideal $L^1(\mathcal{E})$ of $L^1(\mathcal{B})$. Thus, by 1. and the Cohen-Hewitt theorem, $L^1(\mathcal{A})=L^1(\mathcal{A})*L^1(\mathcal{A})^*\subseteq L^1(\mathcal{E})*L^1(\mathcal{E})^*$. On the other hand, if $\xi,\,\eta\in C_c(\mathcal{E})$, $t\in G$:

$$\xi*\eta^*(t)=\int_G\xi(s)\eta^*(s^{-1}t)ds=\int_G\xi(s)\Delta(t^{-1}s)\eta(t^{-1}s)^*ds=\int_G\Delta(t^{-1}s)\xi(s)\eta(t^{-1}s)^*ds\in A_t,$$

because $\xi(s)\eta(t^{-1}s)^*\in E_sE^*_{t^{-1}s}\in\mathcal{A}\bigcap B_t=A_t$, $\forall s,t\in G$. It follows that $\xi*\eta^*\in C_c(\mathcal{A})$, and hence that $L^1(\mathcal{E})*L^1(\mathcal{E})^*\subseteq L^1(\mathcal{A})$.

Consider now $\xi,\,\eta\in C_c(\mathcal{E})$, $t\in G$. We have:

$$\xi^**\eta(t)=\int_G\xi^*(s)\eta(s^{-1}t)ds=\int_G\Delta(s^{-1})\xi(s^{-1})^*\eta(s^{-1}t)ds=\int_G\Delta(s^{-1})\xi(s^{-1})^*\eta(s^{-1}t)ds.$$

Let $\{(f_V,V)\}_{V\in\mathcal{V}}$ be an approximate unit of $L^1(G)$ as in Lemma 4.14. For $\xi\in C_c(\mathcal{E})$, $V\in\mathcal{V}$, $r\in G$, define $\xi_{V,r}:G\to\mathcal{B}$ by $\xi_{V,r}(s)=\Delta(s^{-1})f_V(r^{-1}s^{-1})\xi(s)$. Then $\xi_{V,r}\in C_c(\mathcal{E})$, and we have:

$$\xi^*_{V,r}*\eta(t)=\int_G\Delta(s^{-1})\Delta(s)f_V(r^{-1}s)\xi(s^{-1})^*\eta(s^{-1}t)ds=\int_G f_V(r^{-1}s)\zeta_{\xi,\eta}(s,t)ds,$$

where $\zeta_{\xi,\eta}:G\times G\to\mathcal{B}$ is such that $\zeta_{\xi,\eta}(s,t)=\xi(s^{-1})^*\eta(s^{-1}t)$. Note that $\zeta_{\xi,\eta}$ is continuous of compact support: $\mathrm{supp}(\zeta_{\xi,\eta})\subseteq(\mathrm{supp}(\xi))^{-1}\times(\mathrm{supp}(\xi))^{-1}\mathrm{supp}(\eta)$. By Lemma 4.14, we see that $\lim_V\xi^*_{V,r}*\eta=\zeta_{\xi,\eta,r}$ in the inductive limit topology $C_c(\mathcal{B})$, and hence also in $L^1(\mathcal{B})$. So, we have that $\big(\overline{\mathrm{span}}L^1(\mathcal{E})^**L^1(\mathcal{E})\big)\bigcap C_c(\mathcal{B})\supseteq Z$, where $Z=\mathrm{span}\{\zeta_{\xi,\eta,r}:\ \xi,\eta\in C_c(\mathcal{E}),\,r\in G\}$.

To see that $\overline{\mathrm{span}}L^1(\mathcal{E})^*L^1(\mathcal{E})=L^1(\mathcal{B})$, it is sufficient to see that Z is dense in $C_c(\mathcal{B})$ in the inductive limit topology. By II-14.6 of [14], for this is enough to verify that: (a) $Z(t)$ is dense in B_t, $\forall t\in G$, where $Z(t)=\{\zeta(t):\ \zeta\in Z\}$ and (b) if $g:G\to\mathbb{C}$ is continuous, then $g\zeta\in Z$, $\forall\zeta\in Z$.
(a) We have: $Z(t)\supseteq\{\zeta_{\xi,\eta,r}:\ \xi,\eta\in C_c(\mathcal{E}),r\in G\}=\{\xi(r^{-1})^*\eta(r^{-1}t):\ \xi,\eta\in C_c(\mathcal{E}),r\in G\}$. Therefore, $\overline{Z(t)}\supseteq\overline{\mathrm{span}}\{E^*_{r^{-1}}E_{r^{-1}t}:\ r\in G\}=\overline{\mathrm{span}}\big(B_t\bigcap\mathcal{E}^*\mathcal{E}\big)=B_t$ by the hypothesis in 4.
(b) Let $g:G\to\mathbb{C}$ be a continuous function, $\xi,\,\eta\in C_c(\mathcal{E})$, $r,t\in G$. Defining $g^r:G\to\mathbb{C}$ as $g^r(s)=g(rs)$ we have: $(g\zeta_{\xi,\eta,r})(t)=g(t)\xi(r^{-1})^*\eta(r^{-1}t)=\xi(r^{-1})^*g^r(r^{-1}t)\eta(r^{-1}t)=\zeta_{\xi,g^r\eta,r}(t)$. Since $\zeta_{\xi,g^r\eta,r}\in C_c(\mathcal{E})$, we conclude that $\zeta_{\xi,g^r\eta,r}\in Z$, closing the proof. □

Corollary 4.16. *Let $\mathcal{B}=(B_t)_{t\in G}$ be a Fell bundle, $\mathcal{E}=(E_t)_{t\in G}$ a right ideal of $\mathcal{B}$, and $\mathcal{A}=(A_t)_{t\in G}$ a sub–Fell bundle of $\mathcal{B}$ contained in $\mathcal{E}$. If $\mathcal{A}\mathcal{E}\subseteq\mathcal{E}$ and $\mathcal{E}\mathcal{E}^*\subseteq\mathcal{A}$, we have that $C^*_r(\mathcal{A})$ is a hereditary sub-C^*-algebra of $C^*_r(\mathcal{B})$, and if* span $\big(B_t\bigcap\mathcal{E}^*\mathcal{E}\big)$ *is dense in B_t, for each $t\in G$, then $C^*_r(\mathcal{A})$ and $C^*_r(\mathcal{B})$ are Morita equivalent via the right ideal $C^*_r(\mathcal{E}):=\overline{C_c(\mathcal{E})}\subseteq C^*_r(\mathcal{B})$ of $C^*_r(\mathcal{B})$.*

Proof. By 4.9, $C^*_r(\mathcal{A})$ is naturally isomorphic to the closure of $C_c(\mathcal{A})$ in $C^*_r(\mathcal{B})$. By 4.15 2., $L^1(\mathcal{E})$ is a right ideal of $L^1(\mathcal{B})$, and hence its closure $C^*_r(\mathcal{E})$ in $C^*_r(\mathcal{B})$ is a right ideal of $C^*_r(\mathcal{B})$. Now, it follows from 3. of 4.15 that $C^*_r(\mathcal{A})=C^*_r(\mathcal{E})C^*_r(\mathcal{E})^*$, and therefore $C^*_r(\mathcal{A})$ is a hereditary sub-C^*-algebra of $C^*_r(\mathcal{B})$. Finally, the last assertion follows immediately from 4. of 4.15. □

Corollary 4.17. *Let $\mathcal{B}=(B_t)_{t\in G}$ be a Fell bundle, $\mathcal{E}=(E_t)_{t\in G}$ a right ideal of $\mathcal{B}$, and $\mathcal{A}=(A_t)_{t\in G}$ a sub–Fell bundle of $\mathcal{B}$ contained in $\mathcal{E}$. If $\mathcal{A}\mathcal{E}\subseteq\mathcal{E}$ and $\mathcal{E}\mathcal{E}^*\subseteq\mathcal{A}$, and if* span $\big(B_t\bigcap\mathcal{E}^*\mathcal{E}\big)$ *is dense in B_t, for all $t\in G$, then $\mathcal{B}$ is amenable whenever $\mathcal{A}$ is amenable.*

Proof. Suppose that $\mathcal{A}$ is amenable, and let $\|\cdot\|_{\text{máx}}$ be the norm on $C^*(\mathcal{B})$ and $\|\cdot\|_r$ the norm on $C^*_r(\mathcal{B})$. The closure of $C_c(\mathcal{A})$ in $C^*_r(\mathcal{B})$ is $C^*_r(\mathcal{A})$, by Proposition 4.9. We also have that the closure of $C_c(\mathcal{A})$ in $C^*(\mathcal{B})$ is $C^*(\mathcal{A})=C^*_r(\mathcal{A})$, because any representation of $L^1(\mathcal{B})$ induces a representation of $L^1(\mathcal{A})$ by restriction, and therefore the norm of $C^*(\mathcal{A})$ is greater or equal to $\|\cdot\|_{\max}$. The amenability

of $\mathcal{A}$ implies that these two norms are equal. Let $\xi \in C_c(\mathcal{E})$. Since $C^*(\mathcal{A}) = C^*_r(\mathcal{A})$ by assumption, and $\xi * \xi^* \in C_c(\mathcal{A})$ by 4.15, we have:

$$\|\xi\|_r^2 = \|\xi^* * \xi\|_r = \|\xi * \xi^*\|_r = \|\xi * \xi^*\|_{\text{máx}} = \|\xi^* * \xi\|_{\text{máx}} = \|\xi\|^2_{\text{máx}},$$

and therefore $\|\xi\|_r = \|\xi\|_{\text{máx}}$. Then, the completions of $C_c(\mathcal{E})$ with respect to $\|\cdot\|_{\text{máx}}$ and $\|\cdot\|_r$ agree. Let us denote this completion by E. We have that E is a full Hilbert module over both $C^*(\mathcal{B})$ and $C^*_r(\mathcal{B})$, and hence $C^*(\mathcal{B}) = C^*_r(\mathcal{B})$. This shows that $\mathcal{B}$ is amenable. □

Theorem 4.18. *Let β be a continuous action of G on a C^*-algebra B, $I \lhd B$, $\alpha = \beta\big|_I$. Then $I \rtimes_{\alpha,r} G$ is a hereditary sub–C^*-algebra of $B \rtimes_{\beta,r} G$. If, in addition, $[\beta(I)]$ is dense in B, i.e. β is the enveloping action of α, then $I \rtimes_{\alpha,r} G$ and $B \rtimes_{\beta,r} G$ are Morita equivalent.*

Proof. Let $\mathcal{B}_\beta = (B_t)_{t\in G}$ be the Fell bundle associated with β, $\mathcal{B}_\alpha = (A_t)_{t\in G}$ the Fell bundle associated with α, and $\mathcal{E} = (E_t)_{t\in G}$, where $E_t = \{(t,x) \in \mathcal{B}_\beta : \ x \in I\}$. It is clear that $\mathcal{B}_\alpha \subseteq \mathcal{E} \subseteq \mathcal{B}_\beta$ as Banach bundles, and that $\mathcal{B}_\alpha$ is a sub–Fell bundle of $\mathcal{B}_\beta$. Moreover, if $(r, a_r) \in A_r$, $(s, x_s) \in E_s$, $(t, y_t) \in E_t$, $(u, b_u) \in B_u$, we have:

- $(r, a_r)(s, x_s) = (rs, \alpha_r\big(\alpha_{r^{-1}}(a_r)x_s\big)) \in E_{rs}$, because $\alpha_r\big(\alpha_{r^{-1}}(a_r)x_s\big) \in I$. Therefore: $\mathcal{B}_\alpha\mathcal{E} \subseteq \mathcal{E}$
- $(s, x_s)(t, y_t)^* = (s, x_s)(t^{-1}, \beta_{t^{-1}}(y_t^*)) = (st^{-1}, x_s\beta_{st^{-1}}(y_t^*)) \in A_{st^{-1}}$, because $x_s\beta_{st^{-1}}(y_t^*))$ belongs to $I \bigcap \beta_{st^{-1}}(I) = D_{st^{-1}}$. Consequently, $\mathcal{E}\mathcal{E}^* \subseteq \mathcal{B}_\alpha$.
- $(s, x_s)(u, b_u) = (su, x_s\beta_s(b_u)) \in E_u$ because I is an ideal and $x_s \in I$. Thus, $\mathcal{E}\mathcal{B}_\beta \subseteq \mathcal{E}$.

Thus, we may apply Corollary 4.16, concluding that $I \rtimes_{\alpha,r} G = C^*_r(\mathcal{B}_\alpha)$ is a hereditary sub-C^*-algebra of $C^*_r(\mathcal{B}_\beta) = B \rtimes_{\beta,r} G$.

Suppose now that β is the enveloping action of α, that is, $[\beta(I)]$ is dense in B. To see that $I \rtimes_{\alpha,r} G \overset{M}{\sim} B \rtimes_{\beta,r} G$, it is sufficient to show that $\operatorname{span}\big(\mathcal{E}^*\mathcal{E} \bigcap B_t\big)$ is dense in B_t, for all $t \in G$. Now: $(s, x_s)^*(t, y_t) = (s^{-1}, \beta_{s^{-1}}(x_s^*))(t, y_t) = (s^{-1}t, \beta_{s^{-1}}(x_s^*y_t))$. Therefore $\operatorname{span}(\mathcal{E}^*\mathcal{E} \bigcap B_t) = \{(t, \beta_v(x'y')) : v \in G,\, x', y' \in I\}$. By the Cohen -Hewitt theorem, every $z \in I$ may be written in the form $z = x'y'$, for some $x', y' \in I$. So the set above is exactly $\{t\} \times [\beta(I)]$, which is dense in B_t. □

Corollary 4.19. *If β is the enveloping action of an amenable partial action α (4.7), then β is amenable as well.*

Proof. It follows immediately from 4.17. □

4.3. **An application: dilations of partial representations.** Closing this section, we apply our results to show that any partial representation of an amenable discrete group may be dilated to a unitary representation of the group. Recall from [8] that a partial representation of a discrete group G on the Hilbert space H is a map $u : G \to B(H)$ such that, for $t, s \in G$: (i) $u_e = id_H$; (ii) $u_{t^{-1}} = u_t^*$; (iii) $u_su_tu_{t^{-1}} = u_{st}u_{t^{-1}}$. The conditions above imply that u_t is a partial isometry, and also that $u_{s^{-1}}u_su_t = u_{s^{-1}}u_{st}$, $\forall s, t \in G$. The partial representations of G are in one to one correspondence with the non–degenerate representations of its partial C^*-algebra $C^*_p(G)$, which is constructed as follows. Let $X_t = \{\omega \in 2^G : e, t \in \omega\}$ with the product topology, and $\alpha_t : X_{t^{-1}} \to X_t$ such that $\alpha_t(\omega) = t\omega$, $\forall \omega \in X_{t^{-1}}$. Then α is a partial action on $X := X_e$, and $C^*_p(G)$ is defined to be the corresponding crossed product $C(X) \rtimes_\alpha G$, where we are also denoting by α the partial action induced on $C(X)$.

Proposition 4.20. *Let G be a discrete amenable group, and $u : G \to B(H)$ a partial representation. Then there exist a Hilbert space H^{e}, which contains H as a Hilbert subspace, and a unitary representation $u^{\mathrm{e}} : G \to B(H^{\mathrm{e}})$, such that $u_t = Pu^{\mathrm{e}}_t i$, $\forall t \in G$, where $P : H^{\mathrm{e}} \to H$ is the orthogonal projection of H^{e} on H, and $i : H \to H^{\mathrm{e}}$ is the natural inclusion. In particular, the partial representations of a discrete amenable group are positive definite maps.*

Proof. Let α be the partial action described before. Since G is amenable, we have that $C_p^*(G) = C(X)\rtimes_{\alpha,r} G$. First of all, note that α has an enveloping action α^{e} acting on $X^{\mathrm{e}} = 2^G \setminus \{\emptyset\}$, and given by the same formula as α. Let $\mathcal{B}_\alpha$ be the Fell bundle over G associated with α, 1_t the characteristic function of X_t and, if $a_t \in C(X_t)$, let $a_t\delta_t \in C_c(\mathcal{B}_\alpha)$ be defined as $a_t\delta_t(s) = \delta_{t,s}a_t$, where $\delta_{t,s}$ is the Kronecker symbol. By 6.5 of [8], u defines a unique non–degenerate representation $\pi_u : C_p^*(G) \to B(H)$, such that $\pi_u(1_t\delta_t) = u_t$, $\forall t \in G$. In particular, $\pi_u(1_e\delta_e) = id_H$. By Theorem 4.18, we have that $C_p^*(G)$ is a hereditary sub–C^*-algebra of $C(X^{\mathrm{e}})\rtimes_{\alpha^{\mathrm{e}},r} G$, which is equal to $C(X^{\mathrm{e}})\rtimes_{\alpha^{\mathrm{e}}} G$ because of the amenability of G. Moreover, $C_p^*(G)$ is Morita equivalent to $C(X^{\mathrm{e}})\rtimes_{\alpha^{\mathrm{e}}} G$. Therefore, there exist a Hilbert space H^{e} and a non–degenerate representation $\pi_u^{\mathrm{e}} : C(X_G^{\mathrm{e}})\rtimes_{\alpha^{\mathrm{e}}} G \to B(H^{\mathrm{e}})$, such that H is a Hilbert subspace of H^{e} and π_u is the compression of π_u^{e} to H, i.e.: $\pi_u(x) = P\pi_u^{\mathrm{e}}(x)i$, $\forall x \in C_p^*(G)$, where $P : H^{\mathrm{e}} \to H$ is the orthogonal projection and $i = P^* : H \to H^{\mathrm{e}}$ is the natural inclusion ([14], XI-7.6).

Now, $\pi_u^{\mathrm{e}} = \phi^{\mathrm{e}}\times u^{\mathrm{e}}$, for some covariant representation $(\phi^{\mathrm{e}}, u^{\mathrm{e}})$ of the dynamical system $(C(X^{\mathrm{e}}), \alpha^{\mathrm{e}}, G)$; in particular, $u^{\mathrm{e}} : G \to B(H^{\mathrm{e}})$ is a unitary representation of G.

Note that X is a clopen subset of X^{e}, so $1_e \in C(X^{\mathrm{e}})$, and we may compute, in $C(X^{\mathrm{e}})\rtimes_{\alpha^{\mathrm{e}},r} G$: $(1_e\delta_e)(1\delta_t)(1_e\delta_e) = (1_e\delta_t)(1_e\delta_e) = 1_e\alpha_t^{\mathrm{e}}(1_e)\delta_t = 1_t\delta_t$. Therefore:

$$u_t = \pi_u(1_t\delta_t) = P\pi_u^{\mathrm{e}}(1_t\delta_t)\big|_H = P\pi^{\mathrm{e}}(1_e\delta_e)\pi^{\mathrm{e}}(1\delta_t)\pi^{\mathrm{e}}(1_e\delta_e)\big|_H = P\pi^{\mathrm{e}}(1_e\delta_e)u_t^{\mathrm{e}}\pi^{\mathrm{e}}(1_e\delta_e)\big|_H.$$

Observe now that $\pi^{\mathrm{e}}(1_e\delta_e)$ is an orthogonal projection such that $P\pi^{\mathrm{e}}(1_e\delta_e)\big|_H = \pi(1_e\delta_e) = id_H$, and hence $\pi^{\mathrm{e}}(1_e\delta_e)$ is greater or equal to the orthogonal projection $Q \in B(H^{\mathrm{e}})$ with image H. Thus, we have that: $Q\pi^{\mathrm{e}}(1_e\delta_e) = Q = \pi^{\mathrm{e}}(1_e\delta_e)Q$. On the other hand, it is clear that $PQ = P$, $Q = Qi$, and consequently:

$$P\pi^{\mathrm{e}}(1_e\delta_e) = (PQ)\pi^{\mathrm{e}}(1_e\delta_e) = P\big(Q\pi^{\mathrm{e}}(1_e\delta_e)\big) = PQ = P,$$

$$\pi^{\mathrm{e}}(1_e\delta_e)i = \pi^{\mathrm{e}}(1_e\delta_e)(Qi) = \big(\pi^{\mathrm{e}}(1_e\delta_e)Q\big)i = Qi = i.$$

It follows that $u_t = P\pi^{\mathrm{e}}(1_e\delta_e)u_t^{\mathrm{e}}\pi^{\mathrm{e}}(1_e\delta_e)i = Pu_t^{\mathrm{e}}i$. □

5. Morita equivalence of partial actions and Morita enveloping actions

We have seen previously that there exist partial actions on C^*-algebras that have no enveloping actions. The aim of this section is to introduce a weaker notion of enveloping action, so that any partial action has a unique "weak" enveloping action, and Theorem 4.18 is still valid.

To this end we define and study Morita equivalence of partial actions, and show that the reduced crossed products of Morita equivalent partial actions are Morita equivalent. Then we define the so called Morita enveloping actions. If α is a partial action, we say that β is an enveloping action up to Morita equivalence of α, or just a Morita enveloping action of α, if β is the enveloping action of a partial action that is Morita equivalent to α. For this notion we have a result analogous to 4.18: Proposition 5.17. The investigation of the existence and uniqueness of Morita enveloping actions is postponed until Section 7.

5.1. **Hilbert modules and C^*–ternary rings.** In the next subsection we will introduce the Morita equivalence between partial actions, and to do this it will be convenient to use C^*–ternary rings (C^*-trings for short). It would be possible to use just Hilbert bimodules, but we prefer to view a Hilbert module not as a space where a C^*-algebra is acting on, but rather as an object that has almost the status of a C^*-algebra. So, we will quickly see now some basic facts about C^*-trings that will be needed later.

Let us suppose that $(E, \langle\cdot,\cdot\rangle)$ is a full right Hilbert B-module, and let $A = \mathcal{K}(E)$ be the corresponding C^*-algebra of compact operators, that is, the ideal of $\mathcal{L}(E)$ generated by $\{\theta_{x,y} : x, y \in E\}$, where $\theta_{x,y}(z) = x\langle y, z\rangle_r$. Defining $\langle x, y\rangle_l = \theta_{x,y}$, we have that E is a full left Hilbert A-module, and that E is an $(A - B)$-bimodule that satisfies $\langle x, y\rangle_l z = x\langle y, z\rangle_r$, $\forall x, y, z \in E$; we will say that E is a full Hilbert $(A - B)$-bimodule. So, defining $(x, y, z) = x\langle y, z\rangle_r$, we have a ternary product $(\cdot,\cdot,\cdot)$ on E that relates the actions of A and B on E, and also the left and right inner products on E. The object $\big(E, (\cdot,\cdot,\cdot)\big)$ is

a C^*-tring, and determines the pairs $(A, \langle\cdot,\cdot\rangle_l)$ and $(B, \langle\cdot,\cdot\rangle_r)$ up to isomorphisms. This fact was proved in [30], where the notion of C^*-tring was introduced. Let us recall the exact definitions.

Definition 5.1. *A *-ternary ring is a complex linear space E with a transformation $\mu : E\times E\times E \to E$, called ternary product on E, such that μ is linear in the odd variables and conjugate linear in the second one, and such that: $\mu\big(\mu(x,y,z),u,v\big) = \mu\big(x,\mu(u,z,y),v\big) = \mu\big(x,y,\mu(z,u,v)\big)$, $\forall x,y,z,u,v \in E$. A homomorphism $\phi : (E,\mu) \to (F,\nu)$ of *-ternary rings is a linear transformation from E to F such that $\nu\big(\phi(x),\phi(y),\phi(z)\big) = \phi\big(\mu(x,y,z)\big)$, $\forall x,y,z \in E$. We will write (x,y,z) instead of $\mu(x,y,z)$.*

*A C^*-norm on a *-ternary ring E is a norm such that $\|(x,y,z)\| \le \|x\|\,\|y\|\,\|z\|$ and $\|(x,x,x)\| = \|x\|^3$, $\forall x,y,z \in E$. We then say that $(E,\|\cdot\|)$ is a pre-C^*-tring, and that it is a C^*-tring if it is complete.*

A representation of a $*$-ternary ring (E,μ) on the Hilbert spaces H and K is a homomorphism $\pi : E \to B(H,K)$, where in the last space we consider the ternary product given by $(R,S,T) = RS^*T$. $\pi(E)$ is called a ternary ring of operators, or just TRO.

C^*-trings and TRO's were studied by Zettl in [30]. He proved that every C^*-tring (E,μ) may be uniquely decomposed as a direct sum $E = E^+ \oplus E^-$, where $(E^+, \mu\big|_{E^+})$ and $(E^-, -\mu\big|_{E^-})$ are isomorphic to closed TRO's. We will say that a C^*-tring E is *positive* if $E = E^+$. By a result of Zettl's ([30], 3.2), positive C^*-trings correspond exactly to full Hilbert bimodules, that is, there exist, up to isomorphisms, unique pairs $(E^l, \langle\cdot,\cdot\rangle_l)$ and $(E^r, \langle\cdot,\cdot\rangle_r)$ such that E is a full Hilbert $(E^l - E^r)$-bimodule, and $\langle x,y\rangle_l z = \mu(x,y,z) = x\langle y,z\rangle_r$, $\forall x,y,z \in E$.

Proposition 5.2. *Let $\pi : E \to F$ be a homomorphism of *-ternary rings (5.1), where E and F are C^*-trings. Then π is a contraction, and there exists a unique homomorphism $\pi^r : E^r \to F^r$ such that $\pi^r(\langle x,y\rangle_r) = \langle \pi(x),\pi(y)\rangle_r$, $\forall x,y \in E$. Consequently, we have that $\pi(xb) = \pi(x)\pi^r(b)$, $\forall x \in E$, $b \in E^r$. If π is injective (surjective, an isomorphism), then π is isometric, and π^r is injective (respectively surjective, an isomorphism).*

Proof. If π^r exists, it must be $\pi^r(\langle x,y\rangle_r) = \langle \pi(x),\pi(y)\rangle_r$, $\forall x,y \in E$. Therefore we must see that $\sum_i \langle x_i,y_i\rangle_r = 0$ implies that $\sum_i \langle \pi(x_i),\pi(y_i)\rangle_r = 0$. Now, if $\sum_i \langle x_i,y_i\rangle_r = 0$:

$$\big(\sum_i \langle \pi(x_i),\pi(y_i)\rangle_r\big)^*\big(\sum_j \langle \pi(x_j),\pi(y_j)\rangle_r\big) = \sum_i \langle \pi(y_i), \pi(x_j \sum_j \langle x_j,y_j\rangle_r)\rangle_r = 0.$$

Thus, we have a homomorphism of *-algebras $\pi^r : \operatorname{span}\langle E,E\rangle_r \to F^r$. Since $\operatorname{span}\langle E,E\rangle_r$ is a dense *-ideal of E^r, then π^r has a unique extension to a homomorphism $\pi^r : E^r \to F^r$ ([14], VI-19.11).

Now, if $x \in E$: $\|\pi(x)\|^2 = \|\langle \pi(x),\pi(x)\rangle_r\| = \|\pi^r(\langle x,x\rangle_r)\| \le \|\langle x,x\rangle_r\| = \|x\|^2$, and hence π is a contraction. In particular, π is continuous, and therefore $\pi(xb) = \pi(x)\pi^r(b)$, $\forall x \in E$, $b \in E^r$, because this is true for each $b \in \operatorname{span}\langle E,E\rangle_r$, a dense subset of E^r.

If π is surjective, it is clear that so is π^r. Suppose that π is injective, and let $b \in \ker \pi^r$. Then $\pi(xb) = 0$, $\forall x \in E$, and hence $xb = 0$, $\forall x \in E$, and therefore $\operatorname{span}\langle E,E\rangle_r b = 0$. It follows that $E^r b = 0$, and hence that $b = 0$. Consequently π^r is injective, and therefore isometric. Thus: $\|\pi(x)\|^2 = \|\langle \pi(x),\pi(x)\rangle_r\| = \|\pi^r(\langle x,x\rangle_r)\| = \|\langle x,x\rangle_r\| = \|x\|^2$, so π is isometric. □

Zettl's results together with Proposition 5.2 imply that, up to the fact that E^r is determined up to isomorphisms, we have a functor $E \longmapsto E^r$, $(E \xrightarrow{\pi} F) \longmapsto (E^r \xrightarrow{\pi^r} F^r)$ from the category of C^*-trings to the category of C^*-algebras. Of course, we have a left version of this fact.

Definition 5.3. *Let E be a C^*-tring, and $F \subseteq E$ a closed subspace. We say that F is an ideal of E iff $(E,E,F) \subseteq F$ and $(F,E,E) \subseteq F$). We write $F \lhd E$ to indicate that F is an ideal of E and $\mathcal{I}(E)$ to denote the set of ideals of E.*

It is not hard to see that $F \lhd E$ iff $(E,F,E) \subseteq F$. Let us suppose that E is a positive C^*-tring, so that E is a full Hilbert $(E^l - E^r)$bimodule. Then it is easy to see that F is an ideal of E iff F is a closed sub-$(E^l - E^r)$-bimodule of E. Therefore, rephrasing a well known result in our context, we have (see for instance [26], 3.22):

Proposition 5.4. *Let E be a positive C^*-tring. Then the correspondence $F \longmapsto F^r = \overline{\operatorname{span}}\langle F, F\rangle_r$ is a lattice isomorphism between $\mathcal{I}(E)$ and $\mathcal{I}(E^r)$, with inverse $I \longmapsto EI$. Similarly, the correspondence $F \longmapsto F^l = \overline{\operatorname{span}}\langle F, F\rangle_l$ is a lattice isomorphism between $\mathcal{I}(E)$ and $\mathcal{I}(E^l)$, with inverse $J \longmapsto JE$. Consequently, there is a lattice isomorphism, called the Rieffel correspondence, $\mathsf{R} : \mathcal{I}(E^r) \to \mathcal{I}(E^l)$, such that $\mathsf{R}(I) = \overline{\operatorname{span}}\langle EI, EI\rangle_l$; its inverse is $\mathsf{R}:\ \mathcal{I}(E^l) \to \mathcal{I}(E^r)$ given by $\mathsf{R}(J) = \overline{\operatorname{span}}\langle JE, JE\rangle_r$.*

Corollary 5.5. *Let $\pi : E \to F$ be a homomorphism of *-ternary rings between C^*-trings E and F. Then $\big(\ker(\pi)\big)^r = \ker(\pi^r)$. In particular, π is injective iff π^r is injective.*

5.2. **Morita equivalent partial actions.**

Definition 5.6. *Let E be a positive C^*-tring and $\alpha = (\{E_t\}_{t\in G}, \{\alpha_t\}_{t\in G})$ a set theoretic partial action on E, where $E_t \lhd E$, and $\alpha_t : E_{t^{-1}} \to E_t$ is an isomorphism of C^*-trings, $\forall t \in G$. We say that α is a partial action of G on E if $\{E_t\}_{t\in G}$ is a continuous family (3.1) and, if $\mathcal{E}^{-1} = \{(t, x) :\ x \in E_{t^{-1}}\} \subseteq G \times E$ with the product topology, then the map $\mathcal{E}^{-1} \to E$ such that $(t, x) \longmapsto \alpha_t(x)$ is continuous.*

Proposition 5.7. *Let $\alpha = (\{E_t\}_{t\in G}, \{\alpha_t\}_{t\in G})$ be a partial action of the discrete group G on the C^*-tring E, and consider the pairs: $\alpha^l = (\{E^l_t\}_{t\in G}, \{\alpha^l_t\}_{t\in G})$ and $\alpha^r = (\{E^r_t\}_{t\in G}, \{\alpha^r_t\}_{t\in G})$. Then α^l is a partial action of G on E^l, and α^r is a partial action of G on E^r.*

Proof. By 5.4 and 5.2, $E^r_t \lhd E^r$, $\alpha^r_t : E^r_{t^{-1}} \to E^r_t$ is an isomorphism, $\forall t \in G$, and $\alpha^r_e = id_{E^r}$. It follows from 5.4 that, since α_{st} is an extension of $\alpha_s\alpha_t$, then α^r_{st} is an extension of $(\alpha_s\alpha_t)^r$. Now, the domain of $\alpha_s\alpha_t$ is $E_{t^{-1}} \bigcap E_{t^{-1}s^{-1}}$, and $\alpha_s\alpha_t : E_{t^{-1}} \bigcap E_{t^{-1}s^{-1}} \to E_t \bigcap E_{st}$ is an isomorphism. By 5.4, the domain of $(\alpha_s\alpha_t)^r$ is $E^r_{t^{-1}} \bigcap E^r_{t^{-1}s^{-1}}$. Since α is a partial action, we also have by 5.4 that $\alpha^r_t\big(E^r_{t^{-1}} \bigcap E^r_r\big) = E^r_t \bigcap E^r_{tr}$. It follows that the domain of $\alpha^r_s\alpha^r_t$ is $E^r_{t^{-1}} \bigcap E^r_{t^{-1}s^{-1}}$, and hence that $\alpha^r_s\alpha^r_t = (\alpha_s\alpha_t)^r$. Therefore, α^r_{st} is an extension of $\alpha^r_s\alpha^r_t$. ∎

Remark 5.8. It is clear that any partial action γ on a C^*-tring E is also a partial action on E^*, the adjoint C^*-tring of E (By E^* we mean the C^*-tring naturally associated to the adjoint bimodule of the Hilbert E^r-module E). Let γ denote this partial action. Then it is easy to see that $(\gamma^*)^l = \gamma^r$, and $(\gamma^*)^r = \gamma^l$.

Example 5.9. If E is a right ideal of a C^*-algebra A, where G acts by an action β, then $\alpha := \beta\big|_E$ is a partial action on E, and we have that $\alpha^r = \beta\big|_{\overline{\operatorname{span}}E^*E}$, and $\alpha^l = \beta\big|_{\overline{\operatorname{span}}EE^*}$.

Definition 5.10. *Let $\alpha = (\{A_t\}_{t\in G}, \{\alpha_t\}_{t\in G})$ and $\beta = (\{B_t\}_{t\in G}, \{\beta_t\}_{t\in G})$ be partial actions of G on the C^*-algebras A and B respectively. We say that α is Morita equivalent to β if there exists a partial action $\gamma = (\{E_t\}_{t\in G}, \{\gamma_t\}_{t\in G})$ on a positive C^*-tring E, such that $\gamma^l = \alpha$, and $\gamma^r = \beta$. We will denote this relation by $\alpha \overset{M}{\sim} \beta$.*

Remark 5.11. In [4], Combes defined Morita equivalence of actions. When in Definition 5.10 α and β are global actions, γ must also be a global action, and therefore our definition of Morita equivalence agrees with his in this case. Note that, in fact, α, β and γ are global actions if and only if one of them is a global action.

Lemma 5.12. *Let E be a C^*-tring and $A = E^r$. If $\{D_t\}_{t\in G}$ is a continuous family of ideals in A, and $E_t := ED_t$, $\forall t \in G$, then $\{E_t\}_{t\in G}$ is a continuous family of ideals in E.*

Proof. Let $U \subseteq E$ be an open set, $G_U = \{s \in G :\ U \cap E_s \neq \emptyset\}$, and $t \in G_U$. Consider $x \in U \cap E_t$. By Cohen–Hewitt, $x = ya$, for some $y \in E$, and $a \in D_t$. Since the action of A on E is continuous, there exist open sets $V \subseteq E$ and $W \subseteq A$, such that $y \in V$, $a \in W$, and $VW \subseteq U$. Now, $a \in W \cap D_t$, and since $\{D_s\}_{s\in G}$ is continuous, the set $G_W = \{s \in G :\ W \cap D_s \neq \emptyset\}$ is open and contains t. For each $s \in G_W$ take $a_s \in W \cap D_s$. Then $xa_s \in E_s \cap VW \subseteq E$, so $t \in G_W \subseteq G_U$, and hence G_U is open. ∎

We will see next that Morita equivalence of partial actions is an equivalence relation. Recall that if E is a $(A-B)$–Hilbert bimodule and F is a $(B-C)$–Hilbert bimodule, their inner tensor product is the $(A-C)$–Hilbert bimodule $E\bigotimes_B F$ constructed as follows: let $E\bigodot F$ their algebraic tensor product, and consider on $E\bigodot F$ the unique C–sesquilinear map $\langle\cdot,\cdot\rangle_r'$ such that $\langle x_1\odot y_1, x_2\odot y_2\rangle = \langle y_1, \langle x_1,x_2\rangle_B y_2\rangle_C$, where $\langle\cdot,\cdot\rangle_C$ is the C–inner product on F, and $\langle\cdot,\cdot\rangle_B$ is the B–inner product on E. This sesquilinear map is a semi–inner product, that defines an inner product on the quotient $(E\bigodot F)/N$, where $N=\{z\in E\bigodot F:\ \langle z,z\rangle_r'=0\}=\operatorname{span}\{xb\odot y-x\odot by:\ x\in E, y\in F, b\in B\}$. Then, $E\bigotimes_B F$ is the completion of $(E\bigodot F)/N$ with respect to this inner product (see for instance [16] for details). We will denote by $x\otimes y$ the projection of $x\odot y$ on $E\bigotimes_B F$.

Lemma 5.13. *Let $\mu_i: E_i\to F_i$ be C^*-trings homomorphisms for $i=1,2$. Suppose that A, B and C are C^*-algebras such that $E_1^l, F_1^l\lhd A$, $E_2^r, F_2^r\lhd C$, and $E_1^r=E_2^l\lhd B$. Suppose, moreover, that $\mu_1^r=\mu_2^l$. Then there exists a unique homomorphism of C^*-trings $\mu_1\otimes_B\mu_2: E_1\bigotimes_B E_2\to F_1\bigotimes_B F_2$ such that $(\mu_1\otimes_B\mu_2)(x_1\otimes x_2)=\mu_1(x_1)\otimes\mu_2(x_2)$, $\forall x_1\in E_1,\ x_2\in E_2$. If μ_1 and μ_2 are isomorphisms, then $\mu_1\otimes_B\mu_2$ also is an isomorphism. Moreover, we have that $(\mu_1\otimes_B\mu_2)^l=\mu_1^l$, and $(\mu_1\otimes_B\mu_2)^r=\mu_2^r$.*

Proof. Let $\mu_1\odot\mu_2: E_1\bigodot E_2\to F_1\bigodot F_2$ be the unique linear transformation such that $x_1\odot x_2\longmapsto \mu_1(x_1)\odot\mu_2(x_2)$, $\forall x_1\in E_1,\ x_2\in E_2$; it is a homomorphism of *–ternary rings. Let $\langle\cdot,\cdot\rangle$ and $[\cdot,\cdot]$ be the corresponding C–pre–inner products on $E_1\bigodot E_2$ and $F_1\bigodot F_2$ respectively. Pick $z,z'\in E_1\bigodot E_2$, $z=\sum_i x_i\odot y_i$, $z'=\sum_j x_j'\odot y_j'$. Since $[(\mu_1\odot\mu_2)(z),(\mu_1\odot\mu_2)(z')]=\sum_{i,j}\langle\mu_2(y_i),\mu_1^r(\langle x_i,x_j'\rangle_B^{E_1})\mu_2(y_j')\rangle_C^{F_2}$, $\mu_2^r(\langle z,z'\rangle)=\sum_{i,j}\langle\mu_2(y_i),\mu_2^l(\langle x_i,x_j'\rangle_B^{E_1})\mu_2(y_j')\rangle_C^{F_2}$, and $\mu_1^r=\mu_2^l$, we conclude that $[(\mu_1\odot\mu_2)(z),(\mu_1\odot\mu_2)(z')]=\mu_2^r(\langle z,z'\rangle)$. By taking $z=z'$ and computing norms, we have: $\|(\mu_1\odot)\mu_2(z)\|^2=\|\mu_2^r(\langle z,z\rangle)\|\leq\|z\|^2$. Thus, $\mu_1\odot\mu_2$ factors through the quotient, where it is a contraction, and hence extends by continuity to a homomorphism of C^*-trings $\mu_1\otimes_B\mu_2: E_1\bigotimes_B E_2\to F_1\bigotimes_B F_2$. We have, $\forall z,z'\in E_1\bigotimes_B E_2$: $[(\mu_1\otimes_B\mu_2)(z),(\mu_1\otimes_B\mu_2)(z')]=\mu_2^r(\langle z,z'\rangle)$, and therefore $(\mu_1\otimes_B\mu_2)^r=\mu_2^r$. Similarly, $(\mu_1\otimes_B\mu_2)^l=\mu_1^l$. Finally, if μ_1, μ_2 are isomorphisms, we apply the first part of the proof to the maps μ_1^{-1} e μ_2^{-1}, and we note that $id_{E_1}\otimes id_{E_2}=id_{E_1\bigotimes_B E_2}$, $id_{F_1}\otimes id_{F_2}=id_{F_1\bigotimes_B F_2}$. □

Proposition 5.14. *Morita equivalence of partial actions is an equivalence relation.*

Proof. The reflexive and symmetric properties are immediately verified (see Remark 5.8).

Suppose now that $\alpha=(\{A_t\},\{\alpha_t\})$ is a partial action of G on A, $\beta=(\{B_t\},\{\beta_t\})$ is a partial action of G on B, and $\gamma=(\{C_t\},\{\gamma_t\})$ is a partial action of G on C, such that $\alpha\overset{M}{\sim}\beta$ through the partial action $\mu=(\{E_t\},\{\mu_t\})$ of G on the C^*-tring E, and $\beta\overset{M}{\sim}\gamma$ through the partial action $\nu=(\{F_t\},\{\nu_t\})$ of G on the C^*-tring F. Consider the family $\mu\otimes_B\nu:=(\{E_t\bigotimes_B F_t\},\{\mu_t\otimes_B\nu_t\})$. Since μ_{rs} extends $\mu_r\mu_s$ and ν_{rs} extends $\nu_r\nu_s$, it follows that $\mu_{rs}\otimes_B\nu_{rs}$ extends $(\mu_r\otimes_B\nu_r)(\mu_s\otimes_B\nu_s)$. It is clear that $E_e\bigotimes_B F_e=E\bigotimes_B F$, $(\mu\otimes_B\nu)_e=id_{E\bigotimes_B F}$. On the other hand: $(E_t\bigotimes_B F_t)^r=C_t$:
$\overline{\operatorname{span}}\langle E_t\bigotimes_B F_t, E_t\bigotimes_B F_t\rangle_C=\overline{\operatorname{span}}\langle F_t,\langle E_t,E_t\rangle_B F_t\rangle_C=\overline{\operatorname{span}}\langle F_t,B_tF_t\rangle_C=\overline{\operatorname{span}}\langle F_t,F_t\rangle_C=C_t$.
Similarly, $(E_t\bigotimes_B F_t)^l=A_t$. Finally, by 5.13, every $\mu_t\otimes_B\nu_t: E_t^{-1}\bigotimes_B F_t^{-1}\to E_t\bigotimes_B F_t$ is an isomorphism, and $(\mu_t\otimes_B\nu_t)^l=\mu_t^l=\alpha_t$, $(\mu_t\otimes_B\nu_t)^r=\nu_t^r=\gamma_t$, so $\mu\otimes_B\nu$ is a set theoretic partial action on $E\bigotimes_B F$, and $(\mu\otimes_B\nu)^l=\alpha$, $(\mu\otimes_B\nu)^r=\gamma$.

It remains to show that $\mu\bigotimes_B\nu$ is continuous. First, note that the family $\{E_t\bigotimes_B F_t\}_{t\in G}$ is continuous by 5.12, because γ is a partial action. Let $\mathcal{E}^{-1}\bigotimes_B\mathcal{F}^{-1}=\{(t,z):\ z\in E_{t^{-1}}\bigotimes_B F_{t^{-1}}\}$. Note that if $f\in C_c(\mathcal{E}^{-1})$, $g\in C_c(\mathcal{F}^{-1})$, then $f\oslash_B g: G\to\mathcal{E}^{-1}\bigotimes_B\mathcal{F}^{-1}$ such that $(f\oslash_B g)(t)=\big(t,f(t)\otimes g(t)\big)$ is a continuous section of the Banach bundle $\mathcal{E}^{-1}\bigotimes_B\mathcal{F}^{-1}$, and that for each $t\in G$, $\operatorname{span}\{(f\oslash_B g)(t):\ f\in C_c(\mathcal{E}^{-1}), g\in C_c(\mathcal{F}^{-1})\}$ is dense in $(\mathcal{E}^{-1}\bigotimes_B\mathcal{F}^{-1})_t$. On the other hand, the map $G\to G\times(E\bigotimes_B F)$ such that $t\longmapsto\big(t,\mu_t\big(f(t)\big)\otimes\nu_t\big(g(t)\big)\big)$ is continuous, $\forall f\in C_c(\mathcal{E}^{-1})$, $g\in C_c(\mathcal{F}^{-1})$. So we conclude, by [14], II-14.6 and II-13.16, that the application $\mathcal{E}^{-1}\bigotimes_B\mathcal{F}^{-1}\to G\times(E\bigotimes_B F)$ such that $(t,x)\longmapsto(t,(\mu_t\otimes\nu_t)(x))$ is a continuous homomorphism of Banach bundles, and therefore $\mu\otimes_B\nu$ is a continuous partial action. Thus $\alpha\overset{M}{\sim}\gamma$. □

Proposition 5.15. *Let $\alpha = (\{A_t\}_{t\in G}, \{\alpha_t\}_{t\in G})$ and $\beta = (\{B_t\}_{t\in G}, \{\beta_t\}_{t\in G})$ be partial actions of G on the C^*-algebras A and B respectively. If α and β are Morita equivalent, then $A \rtimes_{\alpha,r} G \overset{M}{\sim} B \rtimes_{\beta,r} G$.*

Proof. Suppose that α and β are Morita equivalent through a partial action $\gamma = (\{E_t\}, \{\gamma_t\})$; say that $\gamma^l = \alpha$, and $\gamma^r = \beta$. Consider the full left Hilbert A–module $A \bigoplus E$, where $\langle (a_1, e_1), (a_2, e_2)\rangle = a_1a_2^* + \langle e_1, e_2\rangle_l$. Then $A \bigoplus E$ establishes a Morita equivalence between A and $\mathbb{L}(E)$, the linking algebra of E (see [26] for instance). By 5.4, every ideal A_t corresponds to an ideal $\mathsf{R}(A_t) \lhd \mathbb{L}(E)$, and it is easy to see that $\mathsf{R}(A_t) = \mathbb{L}(E_t)$, $\forall t \in G$. Since $\mathbb{L}(E_t) = \begin{pmatrix} A_t & E_t \\ E_t^* & B_t \end{pmatrix}$, and (A_t), (E_t), (E_t^*) and (B_t) are continuous families, so it is the family $(\mathbb{L}(E_t))_{t\in G}$. Let us define now $\mathbb{L}(\gamma_t) : \mathbb{L}(E_{t^{-1}}) \to \mathbb{L}(E_t)$ by $\mathbb{L}(\gamma_t) : \begin{pmatrix} a_{t^{-1}} & x_{t^{-1}} \\ y_{t^{-1}} & b_{t^{-1}} \end{pmatrix} \longmapsto \begin{pmatrix} \alpha_t(a_{t^{-1}}) & \gamma_t(x_{t^{-1}}) \\ \gamma_t(y_{t^{-1}}) & \beta_t(b_{t^{-1}}) \end{pmatrix}$. Since $\gamma_t(a_{t^{-1}}x_{t^{-1}}) = \alpha_t(a_{t^{-1}})\gamma_t(x_{t^{-1}})$ and $\gamma_t(x_{t^{-1}}b_{t^{-1}}) = \gamma_t(x_{t^{-1}})\beta_t(b_{t^{-1}})$, $\forall x_{t^{-1}} \in E_{t^{-1}}$, $a_{t^{-1}} \in A_{t^{-1}}$, $b_{t^{-1}} \in B_{t^{-1}}$, we have that $\mathbb{L}(\gamma) = (\{\mathbb{L}(E_t)\}_{t\in G}, \{\mathbb{L}(\gamma_t)\}_{t\in G})$ is a partial action of G on $\mathbb{L}(E)$. We call $\mathbb{L}(\gamma)$ the *linking partial action* of γ. Observe that if γ_1 is the restriction of $\mathbb{L}(\gamma)$ to $\begin{pmatrix} A & E \\ 0 & 0 \end{pmatrix}$, then $\gamma_1^l = \alpha$ and $\gamma_1^r = \mathbb{L}(\gamma)$. Similarly, it is enough to restrict $\mathbb{L}(\gamma)$ to $\begin{pmatrix} 0 & 0 \\ E^* & B \end{pmatrix}$ to see that it is also Morita equivalent to β.

The considerations above show that we may assume that $A = pBp$, for some projection $p \in M(B)$, and that α and γ are the restrictions of β to $pBp = A$ and to $E = pB$ respectively. Let now $\mathcal{A}$ and $\mathcal{B}$ be the Fell bundles corresponding to α and β respectively, and let $\mathcal{E} = (\{t\} \times E_t)_{t\in G}$. We have that $\mathcal{A} \subseteq \mathcal{E} \subseteq \mathcal{B}$, $\mathcal{A}$ is a sub–Fell bundle of $\mathcal{B}$, and that $\mathcal{E}$ is a sub–Banach bundle of $\mathcal{B}$. On the other hand, if $(r, a_r) \in \mathcal{A}$, (s, x_s), $(t, y_t) \in \mathcal{E}$, and $(u, b_u) \in \mathcal{B}$, we have:

- $(r, a_r)(s, x_s) = (rs, \beta_r\big(\beta_r^{-1}(a_r)x_s\big)) \in (rs, \beta_r(A_{r^{-1}}E_s)) \in \mathcal{E}$.
- $(s, x_s)(u, b_u) = (su, \beta_s\big(\beta_s^{-1}(x_s)b_u\big)) \in (su, \beta_s(E_{s^{-1}}B_u)) \in \mathcal{E}$.
- $(s, x_s)(t, y_t)^* = (s, x_s)(t^{-1}, \beta_{t^{-1}}(y_t^*)) = (st^{-1}, \beta_s\big(\beta_{s^{-1}}(x_s)\beta_{t^{-1}}(y_t^*)\big))$, that belongs to $(st^{-1}, \beta_s(A_{s^{-1}} \cap A_{t^{-1}})) \subseteq (st^{-1}, A_s \cap A_{st^{-1}}) \in \mathcal{A}$.
- $(s, x_s)^*(t, y_t) = (s^{-1}, \beta_{s^{-1}}(x_s^*))(t, y_t) = (s^{-1}t, \beta_{s^{-1}}(x_s^*y_t))$. Now, for all $t \in G$, we have that $\overline{\text{span}}\{\beta_{s^{-1}}(x_s^*y_t) : s \in G, x_s \in E_s, y_t \in E_t\} = B_t$: the left member of this equality contains $\overline{\text{span}}E_e^*E_t = \overline{\text{span}}E^*E_t \supseteq \overline{\text{span}}E_t^*E_t = B_t$.

Therefore, we may apply Corollary 4.16, from where we conclude that $A \rtimes_{\alpha,r} G \overset{M}{\sim} B \rtimes_{\beta,r} G$. ☐

This is a good point to introduce the notion of Morita enveloping action.

Definition 5.16. *Let α be a partial action of G on the C^*-algebra A. We say that a continuous action β of G on a C^*-algebra B is a Morita enveloping action of α, if there exists an ideal $I \lhd B$ such that $[\beta(I)]$ is dense in B and $\alpha \overset{M}{\sim} \beta\big|_I$. In other words: β is the enveloping action of a partial action that is Morita equivalent to α.*

We close the section with a result that is similar to Theorem 4.18.

Proposition 5.17. *Let (α, A) be a partial action of G, and assume that (β, B) is a Morita enveloping action of α. Then: $A \rtimes_{\alpha,r} G \overset{M}{\sim} B \rtimes_{\beta,r} G$.*

Proof. Since Morita equivalence of C^*-algebras is transitive, the proof follows immediately by combining Proposition 5.15 above with Theorem 4.18. ☐

6. C^*-algebras of kernels associated with a Fell bundle

In the present section we study two C^*-algebras, $\Bbbk(\mathcal{B})$ and $\Bbbk_r(\mathcal{B})$, that are naturally associated with a given Fell bundle $\mathcal{B}$ over the group G. Both of them are completions of a certain *-algebra of integral operators. The first one is defined by a universal property; the second one is a concrete C^*-algebra of adjointable operators on a Hilbert module. Indeed, using the results of Section 3 we are able to show

that these two C^*-algebras actually agree. There is a natural action of the group on $\Bbbk(\mathcal{B})$. If $\mathcal{B}$ is the Fell bundle of a partial action, we will see in the following section that this natural action on $\Bbbk(\mathcal{B})$ is a Morita enveloping action of the given partial action.

Let $\mathcal{B} = (B_t)_{t\in G}$ be a Fell bundle. Consider a continuous function $k : G \times G \to \mathcal{B}$ of compact support, such that $k(r,s) \in B_{rs^{-1}}$, $\forall r,s \in G$. Such a function will be called a *kernel of compact support* associated with $\mathcal{B}$. The linear space of kernels of compact support associated with $\mathcal{B}$ will be denoted by $\Bbbk_c(\mathcal{B})$ We will see later that any $k \in \Bbbk_c(\mathcal{B})$ may be seen as an integral operator, which justifies this terminology.

Proposition 6.1. *$\Bbbk_c(\mathcal{B})$ is a normed *-algebra with the involution $k^*(r,s) = k(s,r)^*, \forall k \in \Bbbk_c(\mathcal{B})$, the product $k_1 * k_2(r,s) = \int_G k_1(r,t)k_2(t,s)dt$, $\forall k_1,k_2 \in \Bbbk_c(\mathcal{B})$, and the norm $\|k\|_2 = \big(\int_{G^2}\|k(r,s)\|^2 drds\big)^{1/2}$.*

Proof. Let $\nu : G\times G \to G$ be such that $\nu(r,s) = rs^{-1}$, and let $\mathcal{B}_\nu$ be the retraction of $\mathcal{B}$ with respect to ν ([14], II-13.3). Then $\mathcal{B}_\nu$ is a Banach bundle over $G\times G$, and the fiber of $\mathcal{B}_\nu$ over (r,s) is $(r,s,B_{rs^{-1}})$, which we may naturally identify with $B_{rs^{-1}}$. Therefore, $\Bbbk_c(\mathcal{B}) = C_c(\mathcal{B}_\nu)$ as a linear space.

Consider now the map $\mu : G\times G\times G \to \mathcal{B}_\nu$ given by $\mu(t,r,s) = \big(r,s,k_1(r,t)k_2(t,s)\big)$. We have that μ is continuous and has compact support, and that $\mu(t,r,s) \in (\mathcal{B}_\nu)_{(r,s)}$, $\forall t,r,s \in G$. Thus, we may apply [14], II-15.19, from where we conclude that the map $(r,s) \longmapsto \int_G \mu(t,r,s)dt$ is a continuous section of compact support of $\mathcal{B}_\nu$. In other words, $k_1 * k_2 \in \Bbbk_c(\mathcal{B})$.

As for k^*, we have that $\mathrm{supp}(k^*)$ is compact, and $k^*(r,s) = k(s,r)^* \in B^*_{sr^{-1}} = B_{rs^{-1}}$. Consequently, $k^* \in \Bbbk_c(\mathcal{B})$. Routine computations show that $(k_1*k_2)*k_3 = k_1*(k_2*k_3)$, $(k_1*k_2)* = k_2^* * k_1^*$, so $\Bbbk_c(\mathcal{B})$ is a *-algebra. It is also immediate that $\|k^*\|_2 = \|k\|_2$. Finally:

$$\|k_1*k_2\|_2^2 \le \int_G\int_G\left[\int_G \|k_1(r,t)k_2(t,s)\|dt\right]^2 drds \le \int_G\int_G\left[\int_G \|k_1(r,u)\|^2du\int_G\|k_2(v,s)\|^2dv\right]drds$$

□

Proposition 6.2. *Let $\mathcal{B} = (B_t)_{t\in G}$ be a Fell bundle, and consider the action $\Bbbk_c(\mathcal{B})\times C_c(\mathcal{B}) \to C_c(\mathcal{B})$ given by $k\cdot\xi\big|_r = \int_G k(r,s)\xi(s)ds$, $\forall k \in \Bbbk_c(\mathcal{B})$, $\xi \in C_c(\mathcal{B})$, and $r \in G$. With this action, $C_c(\mathcal{B})$ is a $(\Bbbk_c(\mathcal{B}) - B_e)$–bimodule. Moreover, if $\langle\cdot,\cdot\rangle_l : C_c(\mathcal{B})\times C_c(\mathcal{B}) \to \Bbbk_c(\mathcal{B})$ is such that $\langle\xi,\eta\rangle_l\big|_{(r,s)} = \xi(r)\eta(s)^*$, $\forall\xi,\eta \in C_c(\mathcal{B})$, $r,s \in G$, we have:*
1. *$\langle\xi_1,\xi_2\rangle_l\xi_3 = \xi_1\langle\xi_2,\xi_3\rangle_r$, $\forall\xi_1,\xi_2,\xi_3 \in C_c(\mathcal{B})$.*
2. *$\langle k\cdot\xi,\eta\rangle_l = k*\langle\xi,\eta\rangle_l$, $\forall k \in \Bbbk_c(\mathcal{B})$, ξ, $\eta \in C_c(\mathcal{B})$.*
3. *$\langle\xi,\eta\rangle_l^* = \langle\eta,\xi\rangle_l$, $\forall\xi,\eta \in C_c(\mathcal{B})$.*
4. *$\langle\xi,\xi\rangle_l = 0 \iff \xi = 0$.*
5. *$\langle k\xi,\eta\rangle_r = \langle\xi,k^*\eta\rangle_r$, $\forall k \in \Bbbk_c(\mathcal{B})$, ξ, $\eta \in C_c(\mathcal{B})$, where $\langle\cdot,\cdot\rangle_r : C_c(\mathcal{B})\times C_c(\mathcal{B}) \to B_e$ is the right inner product, that is $\langle\xi,\eta\rangle_r = \int_G \xi(s)^*\eta(s)ds$.*

Proof. All these properties are easy to verify. As an example we prove 5., and leave 1.–4. to the reader. If $k \in \Bbbk_c(\mathcal{B})$, ξ, $\eta \in C_c(\mathcal{B})$,

$$\langle k\xi,\eta\rangle_r = \int_G\left[\int_G k(r,s)\xi(s)ds\right]^*\eta(r)dr = \int_G\int_G \xi(s)^*k(r,s)^*\eta(r)drds = \int_G \xi(s)^*(k^*\eta)(s)ds = \langle\xi,k^*\eta\rangle_r.$$

□

We define $I_c(\mathcal{B}) := \mathrm{span}\langle C_c(\mathcal{B}),C_c(\mathcal{B})\rangle_l$, where $\langle\cdot,\cdot\rangle_l$ is the map defined in 6.2. Clearly, $I_c(\mathcal{B})$ is a two–sided *-ideal of $\Bbbk_c(\mathcal{B})$.

Let $E = L^2(\mathcal{B})$, and let $[\cdot,\cdot] : E\times E \to \mathcal{K}(E)$ be the corresponding left inner product (If we think of E as a positive C^*-tring, then $\mathcal{K}(E)$ is nothing but E^l). Note that there is a natural injective *-homomorphism $I_c(\mathcal{B}) \hookrightarrow \mathcal{K}(E)$, the only one such that $\langle\xi,\eta\rangle_l = [\xi,\eta]$, $\forall\xi,\eta \in C_c(\mathcal{B})$. Indeed, if $k = \sum_i\langle\xi_i,\eta_i\rangle_l = 0$, then $k\zeta = 0$, $\forall\zeta \in C_c(\mathcal{B})$, and since $k\zeta = \sum_i[\xi_i,\eta_i]\zeta$, and $C_c(\mathcal{B})$ is dense in E, we see that $\sum_i[\xi_i,\eta_i] = 0 \in \mathcal{K}(E)$. On the other hand, since $\mathrm{span}[C_c(\mathcal{B}),C_c(\mathcal{B})]$ is dense in $\mathcal{K}(E)$, we have

that $\mathcal{K}(E)$ is a C^*-completion of the *-algebra $I_c(\mathcal{B})$. We will see later that this inclusion extends to an inclusion $\Omega : \Bbbk_c(\mathcal{B}) \to \mathcal{L}(E)$ (Theorem 6.9).

Note that, as a Banach space, the completion $HS(\mathcal{B})$ of $(\Bbbk_c(\mathcal{B}), \|\cdot\|_2)$ agrees with $\mathfrak{L}^2(\mathcal{B}_\nu)$ (see [14], II-15.7–15.9). When $\mathcal{B} = (B_t)_{t\in G}$ is the trivial Fell bundle over G with constant fiber $\mathbb{C}$ (that is: the Fell bundle associated with the trivial action of G on $\mathbb{C}$), then $\mathfrak{L}^2(\mathcal{B}_\nu) = L^2(G \times G)$ is naturally identified with the Hilbert–Schmidt operators on $L^2(G)$. Hence we may think of $HS(\mathcal{B})$ as the algebra of "Hilbert–Schmidt operators" on $L^2(\mathcal{B})$.

Definition 6.3. *The universal C^*-algebra of $HS(\mathcal{B})$ will be called the C^*-algebra of kernels of the Fell bundle $\mathcal{B}$, and will be denoted by $\Bbbk(\mathcal{B})$. The closure of $I_c(\mathcal{B})$ in $\Bbbk(\mathcal{B})$ will be denoted by $I(\mathcal{B})$.*

6.1. **Natural action on the kernels.** There is a natural action of G on $\Bbbk_c(\mathcal{B})$: $\beta : G \times \Bbbk_c(\mathcal{B}) \to \Bbbk_c(\mathcal{B})$ such that $\beta_t(k)(r, s) = \Delta(t)k(rt, st)$, where Δ is the modular function on G. We have:

$$\beta_t(k_1 * k_2)\big|_{(r,s)} = \Delta(t)^2 \int_G k_1(rt, vt)k_2(vt, st)dv = \int_G \beta_t(k_1)(r, v)\beta_t(k_2)(v, s)dv = \beta_t(k_1) * \beta_t(k_2)\big|_{(r,s)}$$

$$\big(\beta_t(k^*)\big)\big|_{(r,s)} = \Delta(t)k^*(rt, st) = \Delta(t)k(st, rt)^* = \beta_t(k)(s, r)^* = \beta_t(k)^*\big|_{(r,s)}.$$

This action may be extended to $HS(\mathcal{B})$ and hence to $\Bbbk(\mathcal{B})$: doing $u = rt$, $v = st$ in the integral below:

$$\|\beta_t(k)\|_2^2 = \int_G\int_G \Delta(t)^2\|k(rt, st)\|^2 drds = \int_G\int_G \Delta(t)^{-1}\Delta(t)^{-1}\Delta(t)^2\|k(u, v)\|^2 dudv = \|k\|_2^2.$$

Note that β is a continuous action on $\Bbbk_c(\mathcal{B})$ with the inductive limit topology (recall from 6.1 that $\Bbbk_c(\mathcal{B}) = C_c(\mathcal{B}_\nu)$), and therefore is continuous on $HS(\mathcal{B})$. Since $\Bbbk(\mathcal{B})$ is the universal C^*-algebra of $HS(\mathcal{B})$, β also extends to a continuous action on $\Bbbk(\mathcal{B})$. All these actions will be denoted by β.

Lemma 6.4. *Let B be a Banach bundle over the locally compact space X. Suppose that $\Theta \subseteq C_c(X)$ is dense in $C_c(X)$ in the inductive limit topology, and that $F \subseteq C_c(B)$ is a linear subspace such that $\Theta F \subseteq F$. Then the closure of F in the inductive limit topology is the space: $\{g \in C_c(B) : \ g(x) \in \overline{F(x)}, \forall x \in X\}$. In particular, if $\overline{F(x)} = B_x$, $\forall x \in X$, then F is dense in $C_c(B)$.*

Proof. For $x \in X$, the map $e_x : C_c(B) \to B_x$ such that $f \longmapsto f(x)$ is a continuous linear map in the inductive limit topology . Therefore $e_x(\overline{F}) \subseteq \overline{F(x)}$. Conversely, suppose that $g \in C_c(B)$ is such that $g(x) \in \overline{F(x)}$, $\forall x \in X$. Since X is locally compact, there exists a compact subset K of X whose interior contains supp(g). Now, given $\epsilon > 0$ and $x \in K$, there exists $f_x \in F$ such that $\|g(x) - f_x(x)\| < \epsilon$. Since g, f_x and the norm on B are continuous maps, there exists a precompact open neighborhood V_x of x, such that $\|g(y) - f_x(y)\| < \epsilon$, $\forall y \in V_x$. The family $(V_x)_{x\in K}$ is an open covering of K, so it has a finite subcovering $V_{x_1}, \dots, V_{x_m}$. Let $(\psi_j)_1^m$ be a partition of unity of K subordinated to (V_{x_j}), and let $f = \sum_{j=1}^m \psi_j f_{x_j}$. We have that $f \in C_c(B)$ and $\|g - f\|_\infty < \epsilon$. Therefore, it is enough to show that $f \in \overline{F}$. For this, it is sufficient to show that $\psi f' \in \overline{F}$, $\forall \psi \in C_c(G)$ and $f' \in F$. But, since Θ is dense in $C_c(G)$, there exists $\psi' \in \Theta$ such that $\|\psi - \psi'\|_\infty \leq \varepsilon$, for a given ε, and hence $\|\psi f' - \psi' f'\|_\infty \leq \varepsilon\|f'\|_\infty$. Since $\Theta F \subseteq F$, we have that $\psi' f' \in F$, and therefore $\psi f' \in \overline{F}$. □

Proposition 6.5. *The linear β–orbit of $I_c(\mathcal{B})$ is dense in $\Bbbk_c(\mathcal{B})$ in the inductive limit topology.*

Proof. Recall that $\Bbbk_c(\mathcal{B}) = C_c(\mathcal{B}_\nu)$, where $\mathcal{B}_\nu$ is the retraction of $\mathcal{B}$ with respect to the map $\nu : G\times G \to G$ such that $\nu(r, s) = rs^{-1}$. By Lemma 6.4, it suffices to show that, if Z is the linear β–orbit of $I_c(\mathcal{B})$, then: 1. $Z(r, s)$ is dense in $B_{rs^{-1}}$, $\forall r, s \in G$ and 2. $\forall\varphi_i, \psi_i \in C_c(G)$, and $\forall \zeta \in Z$, the function $(r, s) \longmapsto \sum_i \varphi_i(r)\psi_i(s)\zeta(r, s)$ belongs to Z.

1. If $b \in B_{rs^{-1}}$, by Cohen–Hewitt there exist $b_1 \in B_e$, $b_2 \in B_{rs^{-1}}$ such that $b_1b_2 = b$. There also exist sections $\xi, \eta \in C_c(\mathcal{B})$ such that $\xi(e) = \Delta(r)b_1$, $\eta(sr^{-1}) = b_2^*$. Thus we have: $\beta_{r^{-1}}(\langle\xi, \eta\rangle_l)\big|_{(r,s)} = \Delta(r)^{-1}\xi(rr^{-1})\eta(sr^{-1})^* = \Delta(r)^{-1}\Delta(r)b_1(b_2^*)^* = b$.

2. Let $\varphi,\,\psi\in C_c(G)$, $\xi,\,\eta\in C_c(\mathcal{B})$, $r,s,t\in G$. If $\phi:G\to\mathbb{C}$, let $\phi_t:G\to\mathbb{C}$ be given by $\phi_t(s)=\phi(st)$. Then: $(\varphi\otimes\psi)\big(\beta_t(\langle\xi,\eta\rangle_l)\big)\big|_{(r,s)}=\varphi(r)\psi(s)\Delta(t)\xi(rt)\eta(st)^*=\varphi_{t^{-1}}(rt)\psi_{t^{-1}}(st)\Delta(t)\xi(rt)\eta(st)^*=\beta_t\big(\langle\varphi_{t^{-1}}\xi,\bar\psi_{t^{-1}}\eta\rangle_l\big)\big|_{(r,s)}$, and therefore $(\varphi\otimes\psi)\big(\beta_t(\langle\xi,\eta\rangle_l)\big)\in Z$, because $\varphi_{t^{-1}}\xi,\,\bar\psi_{t^{-1}}\eta\in C_c(\mathcal{B})$. □

Recall that a Fell bundle $\mathcal{B}$ is called *saturated* if for all $r,s\in G$ we have that $\operatorname{span}B_rB_s$ is dense in B_{rs}. The ideal $I_c(\mathcal{B})$ measures the level of saturation of the bundle:

Proposition 6.6. *$\mathcal{B}$ is saturated if and only if $I_c(\mathcal{B})$ is dense in $\Bbbk_c(\mathcal{B})$ in the inductive limit topology.*

Proof. First Suppose that $\mathcal{B}$ is saturated. If $\varphi,\,\psi\in C_c(G)$, $\xi,\,\eta\in C_c(\mathcal{B})$, then $(\varphi\otimes\psi)\langle\xi,\eta\rangle_l=\langle\varphi\xi,\bar\psi\eta\rangle_l\in I_c(\mathcal{B})$. On the other hand, since $\mathcal{B}$ is saturated, given $a\in B_{rs^{-1}}$ and $\epsilon>0$, there exist $b_1,\dots,b_n\in B_r$, $c_1,\dots,c_n\in B_s$, such that $\|b-\sum_{i=1}^n b_ic_i^*\|<\epsilon$. Moreover, there exist sections $\xi_1,\dots,\xi_n,\eta_1,\dots,\eta_n\in C_c(\mathcal{B})$ such that $\xi_j(r)=b_j$ and $\eta_j(s)=c_j$, $\forall j=1,\dots,n$. Therefore: $\|b-\sum_{i=1}^n\langle\xi_i,\eta_i\rangle_l\big|_{(r,s)}\|=\|b-\sum_{i=1}^n\xi_i(r)\eta_i(s)^*\|=\|b-\sum_{i=1}^n b_ic_i^*\|<\epsilon$. It follows from 6.4 that $I_c(\mathcal{B})$ is dense in $\Bbbk_c(\mathcal{B})$ in the inductive limit topology.

Conversely, assume that $\mathcal{B}$ is not saturated: there exist $r,s\in G$ such that $\overline{\operatorname{span}}B_rB_{s^{-1}}\neq B_{rs^{-1}}$. Let $b\in B_{rs^{-1}}$ be such that $b\notin\overline{\operatorname{span}}B_rB_{s^{-1}}$, and let d be the distance from b to $\overline{\operatorname{span}}B_rB_{s^{-1}}$. There exists a continuous section of compact support of $\mathcal{B}_\nu$ that takes the value b in (r,s). In other words, there exists $k\in\Bbbk_c(\mathcal{B})$ such that $k(r,s)=b$. Now, if $\xi_i,\,\eta_i\in C_c(\mathcal{B})$: $\|k-\sum_i\langle\xi_i,\eta_i\rangle_l\|_\infty\geq\|k(r,s)-\sum_i\langle\xi_i,\eta_i\rangle_l(r,s)\|=\|b-\sum_i\xi_i(r)\eta(s)^*\|\geq d$, and therefore $k\notin\overline{I_c(\mathcal{B})}$, because $d>0$. □

6.2. **The reduced C^*-algebra of kernels of a Fell bundle.** Our next goal will be to show that the action of $\Bbbk_c(\mathcal{B})$ on $C_c(\mathcal{B})$ extends to all of $E=L^2(\mathcal{B})$, and, from this that there is an injective *-homomorphism $\Omega:\Bbbk_c(\mathcal{B})\to\mathcal{L}(E)$. To do this, we will need to faithfully represent $\Bbbk_c(\mathcal{B})$ as a *–algebra of operators on a Hilbert space, and also to represent E as a ternary ring of operators. Recall that the right regular representation of G on $L^2(G,H)$ is $\rho:G\times L^2(G,H)\to L^2(G,H)$, such that $\rho_t(x)\big|_r=\Delta(t)^{1/2}x(rt)$, $\forall r,t\in G$, $x\in L^2(G,H)$, where Δ is the modular function on G.

Proposition 6.7. *Let $\pi:\mathcal{B}\to B(H)$ be a representation of the Fell bundle $\mathcal{B}$ on a Hilbert space H. For $k\in\Bbbk_c(\mathcal{B})$, let $\pi_c(k):C_c(G,H)\to C_c(G,H)$ be such that $\pi_c(k)x\big|_r=\int_G\pi\big(k(r,s)\big)x(s)ds$ $\forall x\in C_c(G,H)$. Then we have that $\|\pi_c(k)x\|_2\leq\|k\|_2\,\|x\|_2$, and hence $\pi_c(k)$ extends to an operator $\pi_c(k):L^2(G,H)\to L^2(G,H)$, with $\|\pi_c(k)\|\leq\|k\|_2$. Moreover, $\pi_c:\Bbbk_c(\mathcal{B})\to B\big(L^2(G,H)\big)$ is a bounded representation of the normed *–algebra $\Bbbk_c(\mathcal{B})$ such that $\pi_c\big(\beta_t(k)\big)=\rho_t\pi_c(k)\rho_{t^{-1}}$, where ρ is the right regular representation of G. If $\pi\big|_{B_e}$ is faithful, then so is π_c; if π is non degenerate, then π_c is non–degenerate.*

Proof. Since $x\in C_c(G,H)$ and the map $\mathcal{B}\times H\to H$ such that $(b,h)\longmapsto\pi(b)h$ is continuous ([14], VIII-8.8), we have that the map $G\times G\to H$ such that $(r,s)\longmapsto\pi\big(k(r,s)\big)x(s)$ is continuous, and since it has compact support, we see that $r\longmapsto\int_G\pi\big(k(r,s)\big)x(s)ds$ is a continuous map of compact support from G to H, so in particular it belongs to $L^2(G,H)$. On the other hand:

$$\|\pi_c(k)x\|_2^2=\int_G\|\left[\int_G\pi\big(k(r,s)\big)x(s)ds\right]\|^2dr\leq\int_G\left[\left(\int_G\|\pi\big(k(r,s)\big)\|^2ds\right)^{1/2}\|x\|_2\right]^2dr\leq\|k\|_2^2\,\|x\|_2^2,$$

so $\|\pi_c(k)\|\leq\|k\|_2$. It follows that $\pi_c(k)$ extends to a bounded operator $\pi_c(k):L^2(G,H)\to L^2(G,H)$, and π_c extends by continuity to a bounded linear map $HS(\mathcal{B})\to B\big(L^2(G,H)\big)$. In addition, $\pi_c(k_1*k_2)=\pi_c(k_1)\pi_c(k_2)$: if $x\in C_c(G,H)$, $k_1,k_2\in\Bbbk_c(\mathcal{B})$ we have that

$$\pi_c(k_1*k_2)x\big|_r=\int_G\pi\left[\int_G k_1(r,t)k_2(t,s)dt\right]x(s)ds=\int_G\pi\big(k_1(r,t)\big)\left[\int_G\pi\big(k_2(t,s)\big)x(s)ds\right]dt$$

$$\pi_c(k_1)\pi_c(k_2)x\big|_r=\int_G\pi\big(k_1(r,s)\big)\left[\pi_c(k_2)x\right](t)dt=\int_G\pi\big(k_1(r,t)\big)\left[\int_G\pi\big(k_2(t,s)\big)x(s)ds\right]dt$$

As for the involution, we have that $\pi_c(k^*)=\pi_c(k)^*$. In fact, if $k\in\mathbb{k}_c(\mathcal{B})$, $x,\ y\in C_c(G,H)$:

$$\langle\pi_c(k)x,y\rangle=\int_G\langle\pi_c(k)x(r),y(r)\rangle dr=\int_{G^2}\langle\pi\big(k(r,s)\big)x(s),y(r)\rangle dsdr=\int_{G^2}\langle x(s),\pi\big(k(r,s)^*\big)y(r)\rangle dsdr$$

$$\langle x,\pi_c(k^*)y\rangle=\int_G\langle x(s),\pi_c(k^*)y(s)\rangle ds=\int_G\langle x(s),\int_G\pi\big(k^*(s,r)^*\big)y(r)dr\rangle ds=\int_{G^2}\langle x(s),\pi\big(k(r,s)^*\big)y(r)\rangle dsdr.$$

It follows that $\pi_c:\mathbb{k}_c(\mathcal{B})\to B\big(L^2(G,H)\big)$ is a bounded representation.

Now, if $k\in\mathbb{k}_c(\mathcal{B})$, $x\in C_c(G,H)$, $t,r\in G$:

$$\pi_c\big(\beta_t(k)\big)x\big|_r=\int_G\pi\big(\beta_t(k)(r,s)\big)x(s)ds=\int_G\Delta(t)\pi\big(k(rt,st)\big)x(s)ds=\int_G\pi\big(k(rt,u)\big)x(ut^{-1})du$$

$$\big(\rho_t\pi_c(k)\rho_{t^{-1}}\big)x\big|_r=\int_G\Delta(t)^{1/2}\pi\big(k(rt,u)\big)\rho_{t^{-1}}(x)(s)ds=\int_G\pi\big(k(rt,u)\big)x(ut^{-1})du,$$

and hence $\pi_c\big(\beta_t(k)\big)=\rho_t\pi_c(k)\rho_{t^{-1}}$.

Suppose that $\pi\big|_{B_e}$ is faithful. If $\pi_c(k)=0$, for $k\in\mathbb{k}_c(\mathcal{B})$, we would have that $\int_G\pi\big(k(r,s)\big)x(s)ds=0$ $\forall x\in C_c(G,H)$. In particular, if $(f_V)_{V\in\mathcal{V}}$ is an approximate unit of $L^1(G)$ like in 4.14, and if $h\in H$, choosing $x(s)=f_V(t^{-1}s)h$ and taking limit with respect to V, by 4.14 we have that $\pi\big(k(r,t)\big)h=0$, and therefore that $\pi\big(k(r,s)\big)=0$, $\forall r,s\in G$. Hence π_c is also faithful.

Finally, let us suppose that π is non–degenerate, and let $y\in L^2(G,H)$ such that $\langle\pi_c(k)x,y\rangle=0$, $\forall x\in L^2(G,H)$, $k\in\mathbb{k}_c(\mathcal{B})$. Then $\int_G\langle\pi_c(k)x(r),y(r)\rangle dr=0$, $\forall k\in\mathbb{k}_c(\mathcal{B})$, $\forall x\in C_c(G,H)$. Thus $\int_G\int_G\langle\pi\big(k(r,s)\big)x(s),y(r)\rangle dsdr=0$, $\forall k\in\mathbb{k}_c(\mathcal{B})$, $x\in C_c(G,H)$, so $\langle\pi\big(k(r,s)\big)x(s),y(r)\rangle=0$ almost everywhere in $G\times G$, and therefore $y(r)=0$ almost everywhere in G, because $\{\pi\big(k(r,s)\big)x(s):\ k\in\mathbb{k}_c(\mathcal{B}),(r,s)\in G\times G,x\in C_c(G,H)\}=H$. In fact, since π is non–degenerate, $\pi\big|_{B_e}$ is also non–degenerate, and hence, given $h\in H$, there exist $b\in B_e$ and $h'\in H$, such that $\pi(b)h'=h$. Now there exist $b_1,\ b_2\in B_e$ such that $b=b_1b_2^*$, and sections $\eta_1,\ \eta_2\in C_c(\mathcal{B})$ such that $\eta_i(e)=b_i$, $i=1,2$. Define $k:G\times G\to\mathcal{B}$ such that $k(r,s)=\eta_1(r)\eta_2(s)^*$. It is clear that $k\in\mathbb{k}_c(\mathcal{B})$. Finally, taking $x\in C_c(G,H)$ such that $x(e)=h'$ we have: $\pi\big(k(e,e)\big)x(e)=\pi\big(\eta_1(e)\eta_2(e)^*\big)h'=\pi(b_1b_2^*)h'=\pi(b)h'=h$. Consequently $\eta=0$ in $L^2(G,H)$, and therefore π_c is non–degenerate. □

Lemma 6.8. *Let $\pi:\mathcal{B}\to B(H)$ be a representation of a Fell bundle $\mathcal{B}$. Then there exists a unique representation $\pi_2:L^2(\mathcal{B})\to B\big(H,L^2(G,H)\big)$, such that $\forall\xi\in C_c(\mathcal{B})$ and $h\in H$ we have: $\pi_2(\xi)h\big|_r=\pi\big(\xi(r)\big)h$. If $\pi\big|_{B_e}$ is faithful, then π_2 also is faithful.*

Proof. Let $\xi\in C_c(\mathcal{B})$ and $h\in H$; then:

$$\|(\pi_2\xi)h\|_2^2=\int_G\langle\pi\big(\xi(r)\big)h,\pi\big(\xi(r)\big)h\rangle dr=\langle\pi\left[\int_G\xi(r^*)\xi(r)dr\right]h,h\rangle=\langle\pi(\langle\xi,\xi\rangle_r)h,h\rangle.$$

It follows that $\pi_2\xi\in L^2(G,H)$. On the other hand, since $\pi(\langle\xi,\xi\rangle_r)$ is a positive operator, the equality above implies that $\|\pi_2\xi\|=\|\pi(\langle\xi,\xi\rangle_r)\|^{1/2}$. Thus, $\|\pi_2\xi\|\le\|\xi\|$, and $\|\pi_2\xi\|=\|\xi\|$ if $\pi\big|_{B_e}$ is faithful. It follows that we may extend π_2 to $L^2(\mathcal{B})$, and this extension is an isometry if $\pi\big|_{B_e}$ is faithful.

An easy computation shows that $(\pi_2\xi)^*$ is determined by the formula $(\pi_2\xi)^*x=\int_G\pi\big(\xi(r)^*\big)x(r)dr\in H$. From this we see that $\pi_2(\xi)\pi_2(\eta)^*\pi_2(\zeta)=\pi_2\big(\xi\langle\eta,\zeta\rangle_r\big)$, $\forall\xi,\eta,\zeta\in C_c(\mathcal{B})$. Indeed, if $h\in H$, $r\in G$:

$$\pi_2(\xi)\pi_2(\eta)^*\pi_2(\zeta)h\big|_r=\pi\big(\xi(r)\big)\int_G\pi\big(\eta(s)^*\big)\pi_2(\zeta)h\big|_s ds=\int_G\pi\big(\xi(r)\eta(s)^*\big)\pi\big(\zeta(s)\big)hds$$

$$\pi_2\big(\xi\langle\eta,\zeta\rangle_r\big)h\big|_r=\pi\big(\xi(r)\langle\eta,\zeta\rangle_r\big)h=\int_G\pi\big(\xi(r)\eta(s)^*\zeta(s)\big)hds=\int_G\pi\big(\xi(r)\eta(s)^*\big)\pi\big(\zeta(s)\big)hds$$

Therefore, π_2 is a homomorphism of positive C^*-trings. □

Theorem 6.9. *Let $\pi : \mathcal{B} \to B(H)$ be a representation of the Fell bundle $\mathcal{B}$, and let $\pi_c : \mathbb{k}_c(\mathcal{B}) \to B\big(L^2(G,H)\big)$ and $\pi_2 : L^2(\mathcal{B}) \to B\big(H, L^2(G,H)\big)$ be the representations provided by 6.7 and 6.8 respectively. Then we have (recall that $E = L^2(\mathcal{B})$):*

1. *For all $k \in \mathbb{k}_c(\mathcal{B})$, $b \in B_e$, and ξ, $\eta \in E$:*
   (a) $\pi_2(k\xi) = \pi_c(k)\pi_2(\xi)$.
   (b) $\pi_c(\langle \xi, \eta \rangle_l) = \pi_2(\xi)\pi_2(\eta)^*$
   (c) $\pi_2(\xi b) = \pi_2(\xi)\pi(b)$
   (d) $\pi(\langle \xi, \eta \rangle_r) = \pi_2(\xi)^*\pi_2(\eta)$.
2. *There exists a unique injective *-homomorphism $\Omega : \mathbb{k}_c(\mathcal{B}) \to \mathcal{L}(E)$ such that*
$$\Omega(k)\xi\big|_r = \int_G k(r,s)\xi(s)ds, \ \forall k \in \mathbb{k}_c(\mathcal{B}) \ \text{ and } \ \forall \xi \in C_c(\mathcal{B}).$$
3. *For any representation $\pi : \mathcal{B} \to B(H)$ such that $\pi\big|_{B_e}$ is faithful, there exists an isomorphism $\mu : \overline{\pi_c\big(\mathbb{k}_c(\mathcal{B})\big)} \to \mathcal{L}(E)$ such that the diagram below commutes:*
$$\begin{tikzcd} \mathbb{k}_c(\mathcal{B}) \arrow[rr, "\Omega"] \arrow[dr, "\pi_c"'] & & \mathcal{L}(E) \\ & \pi_c\big(\mathbb{k}_c(\mathcal{B})\big) \arrow[ur, "\mu"'] & \end{tikzcd}$$

Proof. Let us suppose that 1. is true, and let $\pi : \mathcal{B} \to B(H)$ be a representation such that $\pi\big|_{B_e}$ is faithful. Then the representation $\pi_c : \mathbb{k}_c(\mathcal{B}) \to B\big(L^2(G,H)\big)$ extends uniquely to a faithful representation $\tilde{\pi} : \mathbb{k}_\pi(\mathcal{B}) \to B\big(L^2(G,H)\big)$, where $\mathbb{k}_\pi(\mathcal{B})$ is the completion of $\mathbb{k}_c(\mathcal{B})$ with respect to the norm $\|k\|_\pi = \|\pi_c(k)\|$. On the other hand, $\pi_2 : E \to B\big(H, L^2(G,H)\big)$ is an injective homomorphism of C^*-trings. Thus $\sigma : E \to \pi_2(E)$ such that $\sigma(\xi) = \pi_2(\xi)$ is an isomorphism of C^*-trings. Consequently, the application $Ad_\sigma : \mathcal{L}(E) \to \mathcal{L}\big(\pi_2(E)\big)$ such that $T \longmapsto \sigma T \sigma^{-1}$ is an isomorphism of C^*-algebras. Now, by 1.(a) and (d), we have that $\tilde{\pi}(k) \in \mathcal{L}\big(\pi_2(E)\big)$, and hence $Ad_\sigma^{-1}\big(\tilde{\pi}(k)\big) \in \mathcal{L}(E)$. Moreover, if $\xi \in C_c(\mathcal{B})$, $k \in \mathbb{k}_c(\mathcal{B})$:
$$Ad_\sigma^{-1}\big(\pi_c(k)\big)(\xi) = (\sigma^{-1}\pi_c(k)\sigma)\xi = \sigma^{-1}\big(\pi_c(k)\pi_2(\xi)\big) = \sigma^{-1}\big(\pi_2(k\xi)\big) = \sigma^{-1}\big(\sigma(k\xi)\big) = k\xi.$$
This shows that the map $C_c(\mathcal{B}) \to C_c(\mathcal{B})$ such that $\xi \longmapsto k\xi$, may be extended by continuity to an adjointable operator $\Omega(k) : E \to E$. This way we have constructed a map $\Omega : \mathbb{k}_c(\mathcal{B}) \to \mathcal{L}(E)$. On the other hand, if we define $\mu : \overline{\pi_c\big(\mathbb{k}_c(\mathcal{B})\big)} \to \mathcal{L}(E)$ to be the restriction of Ad_σ^{-1} to $\overline{\pi_c\big(\mathbb{k}_c(\mathcal{B})\big)}$, we have that: $\mu\pi_c = \Omega$. Since $\mu\tilde{\pi} : \mathbb{k}_c(\mathcal{B}) \to \mathcal{L}(E)$ is injective, it follows that Ω is injective and that the norm of $k \in \mathbb{k}_c(\mathcal{B})$ in $\mathbb{k}_\pi(\mathcal{B})$ agrees with $\|\Omega(k)\|$. This proves 2. and 3. .

It remains to prove 1. . Let $h \in H$, $r \in G$, $x \in L^2(G,H)$:
(a) $\pi_2(k\xi) = \pi_c(k)\pi_2(\xi)$:
$$\big(\pi_2(k\xi)\big)h\big|_t = \int_G \pi\big(k(t,s)\big)\pi\big(\xi(s)\big)hds = \int_G \pi\big(k(t,s)\big)(\pi_2\xi)h\big|_s ds = \pi_c(k)(\pi_2\xi)h\big|_t.$$
(b)$\pi_c(\langle \xi, \eta \rangle_l) = \pi_2(\xi)\pi_2(\eta)^*$:
$$\pi_c(\langle \xi, \eta \rangle_l)x\big|_t = \int_G \pi\big(\xi(t)\big)\pi\big(\eta(s)^*\big)x(s)ds = \pi\big(\xi(t)\big)\int_G \pi\big(\eta(s)^*\big)x(s)ds = (\pi_2\xi)(\pi_2\eta)^*x\big|_t.$$
(c)$\pi_2(\xi b) = \pi_2(\xi)\pi(b)$: $\pi_2(\xi b)h\big|_r = \pi\big(\xi b(r)\big)h = \pi\big(\xi(r)b\big)h = \pi\big(\xi(r)\big)\pi(b)h\big|_r = \pi_2(\xi)\pi(b)h\big|_r$.
(d)$\pi(\langle \xi, \eta \rangle_r) = \pi_2(\xi)^*\pi_2(\eta)$:
$$\pi_2(\xi)^*\pi_2(\eta)h = \int_G \pi\big(\xi(r)^*\big)\pi_2(\eta)h\big|_r ds = \int_G \pi\big(\xi(r)^*\big)\pi\big(\eta(r)\big)hdr = \int_G \pi\big(\xi(r)^*\eta(r)\big)hdr = \pi(\langle \xi, \eta \rangle_r)h$$
∎

Definition 6.10. *Let $\mathcal{B}$ be a Fell bundle, $E = L^2(\mathcal{B})$, and $\Omega : \Bbbk_c(\mathcal{B}) \to \mathcal{L}(E)$ the homomorphism defined in Theorem 6.9 above. Let $\Bbbk_r(\mathcal{B}) := \overline{\Omega\big(\Bbbk_c(\mathcal{B})\big)} \subseteq \mathcal{L}(E)$. We will say that $\Bbbk_r(\mathcal{B})$ is the reduced C^*-algebra of kernels of the Fell bundle $\mathcal{B}$.*

Remark 6.11. Note that if $\pi : \mathcal{B} \to B(H)$ is a representation such that $\pi\big|_{B_e}$ is faithful, then $\Bbbk_r(\mathcal{B}) \cong \Bbbk_\pi(\mathcal{B})$ naturally ($\Bbbk_\pi(\mathcal{B})$ was defined in the proof of 6.9). In particular, by 6.7) the natural action β on $\Bbbk_c(\mathcal{B})$ extends to $\Bbbk_r(\mathcal{B})$.

By the universal property of $\Bbbk(\mathcal{B})$, there exists a natural epimorphism $\Bbbk(\mathcal{B}) \to \Bbbk_r(\mathcal{B})$, which will be also denoted by Ω, that is the unique homomorphism such that $\Omega(k)\xi\big|_r = \int_G k(r,s)\xi(s)ds$, $\forall k \in \Bbbk_c(\mathcal{B})$, and $\forall \xi \in C_c(\mathcal{B})$. This homomorphism is also a G-morphism with respect to the natural action β. In a while we will see that, in fact, this homomorphism is an isomorphism.

Remark 6.12. Let $k \in \Bbbk_c(\mathcal{B})$ and $f \in C_c(\mathcal{B})$, and consider $h : G \times G \to \mathcal{B}$ such that $h(r,t) = \int_G f(s)k(s^{-1}r,t)ds$. Note that $h(r,t) \in B_{rt^{-1}}$, $\forall r,t \in G$, because for all $r,s,t \in G$, we have that $f(s)k(s^{-1}r,t) \in B_s B_{s^{-1}rt^{-1}} \subseteq B_{rt^{-1}}$. Hence $h \in \Bbbk_c(\mathcal{B})$. Now, considering $\xi \in C_c(\mathcal{B}) \subseteq L^2(\mathcal{B})$, we have: $\Lambda_f \Omega_k \xi\big|_r = \int_G f(s)\Omega_k(\xi)(s^{-1}r)ds = \int_G\int_G f(s)k(s^{-1}r,t)\xi(t)dtds = \int_G h(r,t)\xi(t)dt = \Omega_h\xi\big|_t$. It follows that $C^*_r(\mathcal{B}) \subseteq M\big(\Bbbk_r(\mathcal{B})\big)$. In particular, any representation of $\Bbbk_r(\mathcal{B})$ defines a representation of $C^*_r(\mathcal{B})$.

Corollary 6.13. *Let $I = I(\mathcal{B})$ be the closure of $I_c(\mathcal{B})$ in $\Bbbk(\mathcal{B})$ and $I_r = I_r(\mathcal{B})$ the closure of $\Omega\big(I_c(\mathcal{B})\big)$ in $\Bbbk_r(\mathcal{B})$. Then:*

1. *$(\Bbbk(\mathcal{B}),\beta)$ is the enveloping action of $(I,\beta\big|_I)$.*
2. *$(\Bbbk_r(\mathcal{B}),\beta)$ is the enveloping action of $(I_r,\beta\big|_{I_r})$.*

Proof. Both parts are direct consequences of 6.5. □

We will see next that the natural epimorphism $\Omega : \Bbbk(\mathcal{B}) \to \Bbbk_r(\mathcal{B})$ is an isomorphism. In fact, we will show that the restriction of Ω to the ideal I is an isomorphism $I \cong I_r$, and therefore we must have $\Bbbk(\mathcal{B}) \cong \Bbbk_r(\mathcal{B})$, because of the uniqueness of the enveloping action. The fact that $\Omega\big|_I : I \to I_r$ is an isomorphism will follow from 6.14 and 6.15 below.

Lemma 6.14. *Let $\mathfrak{c} := \overline{\{\sum_{i=1}^n k_i * k_i^* : n \geq 1,\, k_i \in I_c(\mathcal{B})\}} \subseteq \Bbbk_c(\mathcal{B})$, where the closure is taken in the inductive limit topology. Then, if $\xi \in C_c(\mathcal{B})$, we have that $\langle \xi,\xi\rangle_l \in \mathfrak{c} \cap I_c(\mathcal{B})$.*

Proof. Let $\xi \in C_c(\mathcal{B})$. Since $C_0(\mathcal{B})$ is a non–degenerate right Banach module over B_e, the Cohen–Hewitt theorem implies that there exist $\eta \in C_c(\mathcal{B})$, and $b \in B_e$, such that $\xi = \eta b$. Let $\zeta \in C_c(\mathcal{B})$ such that $\zeta(e) = b^*$. The function $G \to B_e$ such that $t \longmapsto \zeta(t)^*\zeta(t) - bb^*$ is continuous and vanishes at $t = e$. Thus, given $\epsilon > 0$, there exists a neighborhood V_ϵ of e, such that if $t \in V_\epsilon$, then $\|\zeta(t)^*\zeta(t) - bb^*\| < \epsilon$. Let $(f_V)_{V\in\mathcal{V}}$ be an approximate unit of $L^1(G)$ as in Lemma 4.14, and for each $V \in \mathcal{V}$ let $\zeta_V = f_V^{1/2}\zeta$. Then $\zeta_V \in C_c(\mathcal{B})$, and $\langle\zeta_V,\zeta_V\rangle_r \to bb^*$, because if $V \subseteq V_\epsilon$:

$$\|\langle\zeta_V,\zeta_V\rangle_r - bb^*\| = \|\int_G f_V(t)\big(\zeta(t)^*\zeta(t) - bb^*\big)dt\| \leq \int_G f_V(t)\|\zeta(t)^*\zeta(t) - bb^*\|dt \leq \epsilon.$$

Now, for each $V \in \mathcal{V}$ consider the kernel $k_V \in I_c(\mathcal{B})$ given by $k_V = \langle \eta, \zeta_V \rangle_l$. If $V \subseteq V_\epsilon$:

$$
\begin{aligned}
\|\big(\langle \xi, \xi \rangle_l - k_V * k_V^*\big)\big|_{(r,s)}\| &= \|\xi(r)\xi(s)^* - \int_G k_V(r,t)k_V^*(t,s)dt\| \\
&= \|\eta(r)b\big(\eta(s)b\big)^* - \int_G k_V(r,t)k_V(s,t)^*dt\| \\
&= \|\eta(r)bb^*\eta(s)^* - \int_G \eta(r)\zeta_V(t)^*\big(\eta(s)\zeta_V(t)^*\big)^*dt\| \\
&= \|\eta(r)bb^*\eta(s)^* - \int_G \eta(r)\zeta_V(t)^*\zeta_V(t)\eta(s)^*dt\| \\
&= \|\eta(r)\big(bb^* - \langle \zeta_V, \zeta_V \rangle_r\big)\eta(s)^*\| \\
&\leq \epsilon \|\eta\|_\infty^2
\end{aligned}
$$

Thus $k_V * k_V^* \to \langle \xi, \xi \rangle_l$ in the inductive limit topology, and hence $\langle \xi, \xi \rangle_l \in \mathfrak{c}$. ☐

Lemma 6.15. *Let $\big(E', (\cdot,\cdot,\cdot)\big)$ be a *–ternary ring, and $\|\cdot\|$, $|\cdot|$ two C^*-norms on E', with $\|\cdot\| \leq |\cdot|$. Let E be the completion of E' with respect to $\|\cdot\|$, and $\langle \cdot,\cdot \rangle_r : E \times E \to E^r$ the corresponding right inner product. If $\operatorname{span}\langle E', E' \rangle$ is an ideal of E^r, then $\|\cdot\| = |\cdot|$.*

Proof. Let F be the completion of E' with respect to $|\cdot|$, and $[\cdot,\cdot]_r : F \times F \to F^r$ the corresponding right inner product. Let $x_1, \ldots, x_n, y_1, \ldots, y_n \in E'$, and assume that $\sum_{i=1}^n \langle x_i, y_i \rangle = 0 \in E^r$. Then, $\forall z \in E'$: $0 = z\sum_{i=1}^n \langle x_i, y_i \rangle_r = \sum_{i=1}^n (z, x_i, y_i) = z\sum_{i=1}^n [x_i, y_i]_r$, and hence $\sum_{i=1}^n [x_i, y_i]_r = 0 \in F^r$, because E' is dense in F. Thus we may define $\psi : \operatorname{span}\langle E', E' \rangle_r \to F^r$ such that $\psi(\sum_{i=1}^n \langle x_i, y_i \rangle) = \sum_{i=1}^n [x_i, y_i]_r$. It is straightforward to check that ψ is a *–homomorphism. Since $\operatorname{span}\langle E', E' \rangle_r$ is an ideal of E^r, it is contractive (by [14], VI-19.11). In consequence we have, $\forall x \in E'$: $|\,x\,|^2 = |\,[x,x]\,| = |\,\psi(\langle x, x \rangle)\,| \leq \|\langle x, x \rangle\| = \|x\|^2 \leq |\,x\,|^2$, and therefore $\|\cdot\| = |\cdot|$ ☐

Proposition 6.16. *Let $\mathcal{B}$ be a Fell bundle. Then $\Omega : \Bbbk(\mathcal{B}) \to \Bbbk_r(\mathcal{B})$ is an isomorphism.*

Proof. Let $\mathfrak{c}$ be the set defined in 6.14, that is, $\mathfrak{c} \subseteq I_c(\mathcal{B})$ is the closed cone of $\Bbbk_c(\mathcal{B})$ generated by the elements of the form $k * k^*$, with $k \in I_c(\mathcal{B})$. Since the convergence in the inductive limit topology implies the convergence in $HS(\mathcal{B})$, and therefore in $\Bbbk(\mathcal{B})$, we have that $\mathfrak{c} \subseteq \Bbbk(\mathcal{B})^+$. Then, by 6.14, $\langle \cdot,\cdot \rangle_l : C_c(\mathcal{B}) \times C_c(\mathcal{B}) \to \Bbbk(\mathcal{B})$ is an inner product. Let $|\cdot|$ be the norm on $C_c(\mathcal{B})$ induced by $\Bbbk(\mathcal{B})$, that is, $|\,\xi\,| = \|\langle \xi, \xi \rangle_l\|$, where we are considering $\langle \xi, \xi \rangle_l$ as an element of $\Bbbk(\mathcal{B})$. This is a C^*–norm on $C_c(\mathcal{B})$, because by completing it with respect to this norm, one obtains a Hilbert module over $\Bbbk(\mathcal{B})$ (see [16]). Let $\|\xi\|$ be the norm of ξ as an element of $L^2(\mathcal{B})$. Recall that $\|\xi\|^2 = \|[\xi, \xi]\|$, where $[\xi, \xi] \in \Bbbk_r(\mathcal{B})$ is the adjointable map given by $[\xi, \xi](\zeta) = \xi \int_G \xi(s)^*\zeta(s)ds$. So we have that $|\,\xi\,| \geq \|\xi\|$, because $\Omega(\langle \xi, \xi \rangle_l) = [\xi, \xi]$. Now, by 6.15 above (with $E' = C_c(\mathcal{B})$) it follows that $\|\cdot\| = |\cdot|$. Therefore $I = E^r = I_r$, and since $(\beta, \Bbbk(\mathcal{B}))$ and $(\beta, \Bbbk_r(\mathcal{B}))$ are the enveloping actions of $\beta\big|_I$ and $\beta\big|_{I_r}$, then $\Bbbk(\mathcal{B}) = \Bbbk_r(\mathcal{B})$, because of the uniqueness of the enveloping action (Theorem 3.8). ☐

In [22], Quigg introduced the notion of *ideal property* of a C^*-algebra: a property $\mathcal{P}$ of a C^*-algebra is ideal if every C^*-algebra has a largest ideal with property $\mathcal{P}$, and if this property is invariant by Morita equivalence and inherited by ideals.

Corollary 6.17. *Let $\mathcal{P}$ be an ideal property. Then B_e has property $\mathcal{P}$ iff $\Bbbk(\mathcal{B})$ has property $\mathcal{P}$. Therefore B_e is liminal, antiliminal, postliminal or nuclear, iff $\Bbbk(\mathcal{B})$ is respectively liminal, antiliminal, postliminal or nuclear.*

Proof. Since B_e and I are Morita equivalent, B_e has property $\mathcal{P}$ if and only if I has property $\mathcal{P}$. On the other hand, by 3.9, I has property $\mathcal{P}$ if and only if $\Bbbk(\mathcal{B})$ has property $\mathcal{P}$. ☐

Before ending this part, we would like to emphasize the functorial nature of $\Bbbk(\mathcal{B})$. Suppose that $\mathcal{A} = (A_t)$ and $\mathcal{B} = (B_t)$ are Fell bundles, and that $\phi : \mathcal{A} \to \mathcal{B}$ is a Fell bundle homomorphism. For instance, ϕ could be a homomorphism induced by a morphism between partial actions. If $k \in \Bbbk_c(\mathcal{A})$, it is clear that $\Bbbk_c(\phi)(k) \in \Bbbk_c(\mathcal{B})$, where $\Bbbk_c(\phi)(k)\big|_{(r,s)} = \phi(k(r,s))$. Moreover, it is easy to check that the resultant map $\Bbbk_c(\phi) : \Bbbk_c(\mathcal{A}) \to \Bbbk_c(\mathcal{B})$ is a homomorphism of *-algebras, which is injective if ϕ is injective. In addition $\|\Bbbk_c(\phi)(k)\|_2 \leq \|k\|_2$, so $\Bbbk_c(\phi)$ extends to a homomorphism $HS(\phi) : HS(\mathcal{A}) \to HS(\mathcal{B})$ of Banach *–algebras. Hence, there is an induced homomorphism of C^*-algebras $\Bbbk(\phi) : \Bbbk(\mathcal{A}) \to \Bbbk(\mathcal{B})$. It is straightforward to verify that $\mathcal{A} \longmapsto \Bbbk(\mathcal{A})$ and $(\mathcal{A} \xrightarrow{\phi} \mathcal{B}) \longmapsto \big(\Bbbk(\mathcal{A}) \xrightarrow{\Bbbk(\phi)} \Bbbk(\mathcal{B})\big)$ is a functor from the category of Fell bundles to the category of C^*-algebras.

If $\phi : \mathcal{A} \to \mathcal{B}$ is injective, then $\Bbbk(\phi) : \Bbbk(\mathcal{A}) \to \Bbbk(\mathcal{B})$ also is injective. To see this, let $\pi : \mathcal{B} \to B(H)$ be a representation such that $\pi\big|_{B_e}$ is faithful. Then $\pi\phi : \mathcal{A} \to B(H)$ is a representation such that $\pi\phi\big|_{A_e}$ is faithful. Since $\Bbbk_r(\mathcal{A}) = \Bbbk(\mathcal{A})$, by 6.9 we have that π and $\pi\phi$ induce unique faithful representations $\tilde{\pi} : \Bbbk(\mathcal{A}) \to B(H)$ and $\widetilde{\pi\phi} : \Bbbk(\mathcal{A}) \to B(H)$, and it is clear that $\widetilde{\pi\phi} = \tilde{\pi}\Bbbk(\phi)$, which implies that $\tilde{\pi}\Bbbk(\phi)$ is injective. Therefore $\Bbbk(\phi)$ is injective.

7. Existence and uniqueness of the Morita enveloping action

In the sequel we prove the main result of the paper: any partial action has a Morita enveloping action, which is unique up to Morita equivalence. To do this, we apply the previous results to the case when the Fell bundle $\mathcal{B}$ is the one associated with a partial action α. We show that the natural action on $\Bbbk(\mathcal{B})$ is a Morita enveloping action of α. More precisely, assume that $\mathcal{B}$ is the Fell bundle of a partial action $\alpha = (\{D_t\}_{t\in G}, \{\alpha_t\}_{t\in G})$. Then define on E the partial action $\gamma = (\{E_t\}_{t\in G}, \{\gamma_t\}_{t\in G})$, where $E_t = ED_t$, and $\gamma_t : E_{t^{-1}} \to E_t$ is such that $\gamma_t(\xi)\big|_r = \Delta(t)^{1/2}\xi(rt)$, for all $\xi \in E_{t^{-1}} \cap C_c(\mathcal{B})$. In Theorem 7.3, we prove that $\gamma^r = \alpha$ and $\gamma^l = \check{\alpha}$, where $\check{\alpha}$ is the natural action of G on $\Bbbk(\mathcal{B})$. It follows that $(\Bbbk(\mathcal{B}), \check{\alpha})$ is a Morita enveloping action of α, and therefore any partial action has a Morita enveloping action. We prove later in Proposition 7.6 that the Morita enveloping action is unique up to Morita equivalence.

Proposition 7.1. *Let $\mathcal{B}$ be a Fell bundle, β the natural action of G on $\Bbbk_c(\mathcal{B})$, and $I_c(\mathcal{B})$ the two–sided *-ideal of $\Bbbk_c(\mathcal{B})$ defined after Proposition 6.2, that is, $I_c(\mathcal{B}) = \operatorname{span}\langle C_c(\mathcal{B}), C_c(\mathcal{B})\rangle_l$. For $t \in G$, consider the two–sided *-ideal $I_t^c = I_c(\mathcal{B}) \bigcap \beta_t(I_c(\mathcal{B}))$ of $\Bbbk_c(\mathcal{B})$. Then the closure of I_t^c in the inductive limit topology is the set*

$$\{k \in \Bbbk_c(\mathcal{B}) :\ k(r,s) \in \overline{(\operatorname{span} B_rB_s^*) \cap (\operatorname{span} B_{rt}B_{st}^*)},\ \forall r,s \in G\}.$$

Proof. Let $k \in I_c(\mathcal{B})$. There exist ξ_i, $\eta_i \in C_c(\mathcal{B})$, such that $k = \sum_i \langle \xi_i, \eta_i\rangle_l$. In particular, $k(r,s) = \sum_i \xi_i(r)\eta_i(s)^* \in \operatorname{span} B_rB_s^*, \forall r,s \in G$. Now, if $k \in I_t^c$, then $\beta_{t^{-1}}(k) \in I_c(\mathcal{B})$, and therefore $\beta_{t^{-1}}(k)(r,s) \in \operatorname{span} B_rB_s^*$, $\forall r,s \in G$, that is, $\Delta(t)^{-1}k(rt^{-1}, st^{-1}) \in \operatorname{span} B_rB_s^*$, $\forall r,s \in G$. Equivalently, $k(r,s) \in \operatorname{span} B_{rt}B_{st}^*$, $\forall r,s \in G$. Let $\Theta = C_c(G) \bigodot C_c(G)$, which is a dense subalgebra of $C_c(G\times G)$ in the inductive limit topology. If ϕ, $\psi \in C_c(G)$, ξ, $\eta \in C_c(\mathcal{B})$, then $(\phi\otimes\psi)\langle \xi,\eta\rangle_l = \langle \phi\xi, \bar{\psi}\eta\rangle_l$, and hence $\Theta I_t^c \subseteq I_t^c$. The result follows now from Lemma 6.4, ∎

Suppose now that $\alpha = (\{D_t\}_{t\in G}, \{\alpha_t\}_{t\in G})$ is a partial action of G on the C^*-algebra A, and let $\mathcal{B}$ be its associated Fell bundle. For each $t \in G$, consider the subset $E_t := \overline{\operatorname{span}} ED_t$ of E. By the Cohen–Hewitt theorem, $E_t = \{\xi a :\ \xi \in E, a \in D_t\}$. By Proposition 5.4, E_t is an ideal of E. If $\xi : G \to \mathcal{B}$ is a section, there exists a unique function $\xi' : G \to A$ such that $\xi'(r) \in D_r$, $\forall r \in G$, and such that $\xi(r) = (r, \xi'(r))$. ξ is continuous with compact support if and only if ξ' is continuous and of compact support. Suppose that $\xi \in C_c(\mathcal{B})$, and $a \in D_t$. Then $\xi a\big|_r = \xi(r)(e,a) = (r,\xi'(r))(e,a) = (r, \alpha_r\big(\alpha_{r^{-1}}(\xi'(r))a\big))$. Since $\alpha_{r^{-1}}(\xi'(r))a \in D_{r^{-1}} \cap D_t$, we have that $\xi a\big|_r \in D_r \cap D_{rt}$, $\forall r \in G$. Thus E_t is the closure of $E_t^c := \{\xi \in C_c(\mathcal{B}) :\ \xi'(r) \in D_r \cap D_{rt}, \forall r \in G\}$.

Proposition 7.2. *Let $\alpha = (\{D_t\}_{t\in G}, \{\alpha_t\}_{t\in G})$ be a partial action of G on A, with associated Fell bundle $\mathcal{B}$. Then:*

1. *$B_rB_s^* = \{(rs^{-1}, D_r \cap D_{rs^{-1}})\}$, $\forall r, s \in G$.*
2. *$B_rB_s^* \cap B_{rt}B_{st}^* = \{(rs^{-1}, D_r \cap D_{rt} \cap D_{rs^{-1}})\}$, $\forall r, s, t \in G$.*
3. *The closures of $\langle E_t^c, E_t^c\rangle_l$ and I_t^c (defined in 7.1) in the inductive limit topology agree, and they are equal to the set: $\{k \in \Bbbk_c(\mathcal{B}) :\ k(r, s) \in B_rB_s^* \cap B_{rt}B_{st}^*,\ \forall r, s \in G\}$*
4. *$E_t^l = I_t$, where I_t is the closure I_t^c in I.*
5. *If $\check{\alpha}$ is the natural action of G on $\Bbbk(\mathcal{B})$, then $\check{\alpha}\big|_I = \big(\{I_t\}_{t\in G}, \{\check{\alpha}_t\big|_{I_t}\}_{t\in G}\big)$.*

Proof. To prove 1., recall that $B_r = (r, D_r)$, $\forall r \in G$. Then

$$\begin{aligned}(r, D_r)(s, D_s)^* &= (r, D_r)(s^{-1}, \alpha_{s^{-1}}(D_s))\\ &= \big(rs^{-1}, \alpha_r\big(\alpha_{r^{-1}}(D_r)\alpha_{s^{-1}}(D_s)\big)\big)\\ &= \big(rs^{-1}, \alpha_r(D_{r^{-1}} \cap D_{s^{-1}})\big)\\ &= (rs^{-1}, D_r \cap D_{rs^{-1}})\end{aligned}$$

2. $B_rB_s^* \cap B_{rt}B_{st}^* = \{(rs^{-1}, D_r \cap D_{rs^{-1}})\} \bigcap \{(rs^{-1}, D_{rt} \cap D_{rs^{-1}})\} = \{(rs^{-1}, D_r \cap D_{rt} \cap D_{rs^{-1}})\}$.
3. Let ξ, $\eta \in E$, $\xi(r) = \big(r, \xi'(r)\big)$ and $\eta(r) = \big(r, \eta'(r)\big)$, $\forall r \in G$. We have:

$$\begin{aligned}\langle \xi, \eta\rangle_l(r, s) &= \xi(r)\eta(s)^*\\ &= \big(r, \xi'(r)\big)\big(s, \eta'(s)\big)^*\\ &= \big(r, \xi'(r)\big)\big(s^{-1}, \alpha_{s^{-1}}(\eta'(s)^*)\big)\\ &= \big(rs^{-1}, \alpha_r\big(\alpha_{r^{-1}}(\xi'(r))\alpha_{s^{-1}}(\eta'(s))\big)\big)\end{aligned}$$

Hence, if ξ, $\eta \in E_t^c$, then

$$\begin{aligned}\langle \xi, \eta\rangle_l \in \big(rs^{-1}, \alpha_r\big(\alpha_{r^{-1}}(D_r \cap D_{rt})\alpha_{s^{-1}}(D_s \cap D_{st})\big)\big)&\\ = \big(rs^{-1}, \alpha_r(D_{r^{-1}} \cap D_t \cap D_{s^{-1}})\big)&\\ = \big(rs^{-1}, D_r \cap D_{rt} \cap D_{rs^{-1}}\big)&\\ = B_rB_s^* \cap B_{rt}B_{st}^*, \quad \text{by 2.}&\end{aligned}$$

Since $B_rB_s^* \cap B_{rt}B_{st}^*$ is a closed linear space, Lemma 6.4 shows that the closure of $\langle E_t^c, E_t^c\rangle_l$ in the inductive limit topology is $\{k \in \Bbbk_c(\mathcal{B}) :\ k(r, s) \in B_rB_s^* \cap B_{rt}B_{st}^*,\ \forall r, s \in G\}$, and by 7.1, this set agrees with the closure of I_t^c in the inductive limit topology. So 3. is proved.
4. This is an immediate consequence of 3.
5. We must show that $I\cap\check{\alpha}_t(I) = I_t$. Since $I_t^c = I_c(\mathcal{B})\cap\check{\alpha}_t(I_c(\mathcal{B})) \subseteq I\cap\check{\alpha}_t(I)$, we have that $I_t \subseteq I\cap\check{\alpha}_t(I)$.

To prove the converse inclusion, by 5.4 it is enough to show that $\big(I\cap\check{\alpha}_t(I)\big)E \subseteq E_t$. Let $x \in I\cap\check{\alpha}_t(I)$. Then $x = \check{\alpha}_t(y)$, for some $y \in I$. For a given $\epsilon > 0$, let $k \in \Bbbk_c(\mathcal{B})$ be such that $\|y - k\| < \epsilon$. Then $\|x - \check{\alpha}_t(y)\| < \epsilon$, because $\check{\alpha}_t$ is an isometry. Consider now $\xi \in C_c(\mathcal{B})$. We have:

$$\check{\alpha}_t(k)\xi\big|_r = \int_G \Delta(t)k(rt, st)\xi(s)ds \in B_{rt}B_{st}B_s.$$

By 1., $B_{rt}B_{st}B_s = (rs^{-1}, D_{rt} \cap D_{rs^{-1}})(s, D_s)$, and therefore

$$\begin{aligned}\check{\alpha}_t(k)\xi\big|_r \in&\big(r, \alpha_{rs^{-1}}(\alpha_{sr^{-1}}(D_{rt} \cap D_{rs^{-1}})D_s)\big)\\ &= \big(r, \alpha_{rs^{-1}}(D_{sr^{-1}} \cap D_{st} \cap D_s)\big)\\ &= (r, D_{rs^{-1}} \cap D_{rt} \cap D_r),\end{aligned}$$

from where it follows that $\check{\alpha}_t(k)\xi \in E_t^c$. By the continuity of the product, it follows now that $xE \subseteq E_t$, $\forall x \in I \cap \check{\alpha}_t(I)$, and this ends the proof. $\square$

Consider now, for each $t \in G$, the map $\gamma_t' : C_c(G, A) \to C_c(G, A)$ such that $\gamma_t'(\xi')\big|_r = \Delta(t)^{1/2}\xi'(rt)$. Suppose that $\xi \in E_{t^{-1}}^c$, where $\xi(r) = \big(r, \xi'(r)\big)$, $\forall r \in G$. Then the map $r \longmapsto (r, \gamma_t'(\xi')(r))$ is an element

of E_t^c: $\gamma_t'(\xi')(r) = \Delta(t)^{1/2}\xi'(rt) \in D_{rt} \cap D_{rtt^{-1}} = D_r \cap D_{rt}$. Therefore, we may define $\gamma_t : E_{t^{-1}}^c \to E_t^c$, such that $\gamma_t(\xi)(r) = (r, \gamma_t'(\xi')(r)) = (r, \Delta(t)^{1/2}\xi'(rt))$.

Theorem 7.3. *Any partial action on a C^*-algebra has a Morita enveloping action. More precisely, let $\alpha = (\{D_t\}_{t\in G}, \{\alpha_t\}_{t\in G})$ be a partial action of G on a C^*-algebra A, and let $\mathcal{B}$ the Fell bundle associated with α. Let E_t^c and γ_t be the ones defined previously. Then:*

1. *For each t, $\gamma_t : E_{t^{-1}}^c \to E_t^c$ is an isometry that extends by continuity to an isomorphism $\gamma_t : E_{t^{-1}} \to E_t$ of C^*-trings.*
2. *$\gamma = (\{E_t\}_{t\in G}, \{\gamma_t\}_{t\in G})$ is a partial action of G on E.*
3. *$\gamma^r = \alpha$, and $\gamma^l = \check{\alpha}\big|_I$, where $\check{\alpha}$ is the natural action of G on $\Bbbk(\mathcal{B})$.*

Proof. Let ξ, $\eta \in C_c(\mathcal{B})$. Identifying A with the unit fiber of $\mathcal{B}$ over the unit element of G, and the kernel $k \in \Bbbk_c(\mathcal{B})$ with $k' : G \times G \to A$ such that $k(r,s) = \big(rs^{-1}, k'(r,s)\big)$, we have the identities:

$$\langle \xi, \eta\rangle_l(r,s) = \alpha_r\big(\alpha_{r^{-1}}(\xi'(r))\alpha_{s^{-1}}(\eta'(s))\big) \tag{1}$$

$$\langle \xi, \eta\rangle_r = \int_G \alpha_{r^{-1}}\big(\xi'(r)^*\eta'(r)\big)dr \tag{2}$$

If ξ, $\eta \in E_{t^{-1}}^c$, by (2) we have that:

$$\begin{aligned}
\langle \gamma_t(\xi), \gamma_t(\eta)\rangle_r &= \int_G \alpha_{r^{-1}}\big(\gamma_t'(\xi')(r)^*\gamma_t'(\eta')(r)\big)dr \\
&= \int_G \alpha_{r^{-1}}\big(\Delta(t)^{1/2}\xi'(rt)^*\Delta(t)^{1/2}\eta'(rt)\big)dr \\
&= \int_G \Delta(t^{-1})\Delta(t)\alpha_{ts^{-1}}\big(\xi'(s)^*\eta'(s)\big)ds \\
&= \int_G \alpha_t\alpha_{s^{-1}}\big(\xi'(s)^*\eta'(s)\big)ds \\
&= \alpha_t(\langle \xi, \eta\rangle_r),
\end{aligned}$$

Then γ_t is an isometry with dense range, and hence it extends to a bijective isometry $\gamma_t : E_{t^{-1}} \to E_t$, such that $\langle \gamma_t(\xi), \gamma_t(\eta)\rangle_r = \alpha_t(\langle \xi, \eta\rangle_r)$, $\forall \xi$, $\eta \in E$. On the other hand, if $\xi \in E_{t^{-1}}^c$ and $a \in D_{t^{-1}}$, we have that $\xi a(r) = (r, \xi'(r))(e,a) = \big(r, \alpha_r(\alpha_{r^{-1}}\xi'(r)a\big)$. Thus:

$$\begin{aligned}
\gamma_t(\xi a)(r) &= \big(r, \Delta(t)^{1/2}\alpha_{rt}\big(\alpha_{rt^{-1}}(\xi'(rt)\big)a\big) \\
&= \big(r, \Delta(t)^{1/2}\alpha_{rt}\big(\alpha_{t^{-1}r^{-1}}(\xi'(rt)\big)\alpha_{t^{-1}}(\alpha_t(a))\big) \\
&= \big(r, \Delta(t)^{1/2}\alpha_r\big(\alpha_{r^{-1}}(\xi'(rt)\big)\alpha_t(a)\big) \\
&= \big(r, \Delta(t)^{1/2}\xi'(rt)\big)\big(e, \alpha_t(a)\big) \\
&= \gamma_t(\xi)\alpha_t(a)(r).
\end{aligned}$$

It follows that γ_t is an isomorphism of C^*-trings, and $\gamma_t^l = \alpha_t$, which proves 1.

Let us see that γ is a set theoretic partial action: it is clear that $E_e = E$ and $\gamma_e = id_E$. Suppose now that $\xi \in E_{t^{-1}}^c$ is such that $\gamma_t(\xi) \in E_{s^{-1}}^c$. Then:

$$\gamma_s\gamma_t(\xi)\big|_r = \big(r, \Delta(s)^{1/2}\big(\gamma_t'(\xi')\big)(rs)\big) = \big(r, \Delta(s)^{1/2}\Delta(t)^{1/2}\xi'(rst)\big) = \big(r, \Delta(st)^{1/2}\xi'(rst)\big) = \gamma_{st}(\xi)\big|_r.$$

Then γ_{st} is an extension of $\gamma_s\gamma_t$, and therefore γ is a set theoretic partial action on E. So, 2. is proved, up to the continuity of γ, that will be proved later.

Since $E^l_t = D_t$ by 5.4, the computation that showed that γ_t was an isometry also implies that $\gamma^r = \alpha$. To see that $\gamma^l = \check{\alpha}\big|_I$, observe that if $\xi,\, \eta \in E^c_{t^{-1}}$, by (1) we have:

$$\begin{aligned}\langle \gamma_t(\xi), \gamma_t(\eta)\rangle_l(r,s) &= \alpha_r\big(\alpha_{r^{-1}}(\gamma'_t(\xi')(r))\alpha_{s^{-1}}(\gamma'_t(\eta')(s))\big)\\ &= \alpha_r\big(\alpha_{r^{-1}}(\Delta(t)^{1/2}\xi'(rt))\alpha_{s^{-1}}(\Delta(t)^{1/2}\eta'(st))\big)\\ &= \Delta(t)\alpha_{rt}\big(\alpha_{rt^{-1}}(\xi'(rt))\alpha_{st^{-1}}(\eta'(st))\big)\\ &= \Delta(t)\langle \xi, \eta\rangle_l(rt, st)\\ &= \check{\alpha}_t(\langle \xi, \eta\rangle_l)(r,s)\end{aligned}$$

Hence $\check{\alpha}_t$ is an extension of γ^l_t, $\forall t \in G$. By Proposition 7.2, we have that $E^l_t = I_t$, $\forall t \in G$, from where we conlude that $\gamma^l = \check{\alpha}\big|_I$. Then 3. is proved.

It remains to show the continuity of γ. We will prove that if ξ_i is a net in E that converges to ξ, with $\xi_i \in E_{t_i^{-1}}$, $\xi \in E_{t^{-1}}$, then $\gamma_{t_i}(\xi_i) \to \gamma_t(\xi)$. Let $\epsilon > 0$. By the Cohen–Hewitt theorem, there exist $\xi' \in E_{t^{-1}}$, $a \in D_{t^{-1}}$, such that $\xi = \xi' a$, and we may suppose that $a \neq 0$. Since $E^c_{t^{-1}}$ is dense in $E_{t^{-1}}$, there exists $\eta \in E^c_{t^{-1}}$ such that $\|\xi' - \eta\| < \epsilon/4\|a\|$. Therefore,

$$\|\xi - \eta a\| \leq \epsilon/4 \tag{3}$$

Since the family $\{D_r\}_{r\in G}$ is continuous, there exists $d \in C_c(G, A)$ such that $d(t^{-1}) = a$. Let us define $\eta_s = \eta d(s^{-1})$. In particular, $\eta_t = \eta a$. Note that $\eta_s \in E^c_{t^{-1}} \bigcap E^c_{s^{-1}}$, because $\eta \in E^c_{t^{-1}}$, and $d(s^{-1}) \in D_{s^{-1}}$, $\forall s \in G$. Since the action of A on E is continuous, and since d is continuous, we have that $\eta_s \to \eta_t$ in E. Therefore, there exists i_1 such that $\forall i \geq i_1$ we have:

$$\|\xi - \eta_{t_i}\| \leq \epsilon/4 \tag{4}$$

(In fact, we even have that $\eta_s \to \eta_t$ in the inductive limit topology, because $\|\eta_s(r) - \eta_t(r)\| = \|\eta(r)d(s^{-1}) - \eta(r)d(t^{-1})\| \leq \|\eta\|_\infty \|d(s^{-1}) - d(t^{-1})\|$).

Let $\eta(r) = \big(r, \eta'(r)\big)$. Since $\eta_s = \eta d(s^{-1})$ we must have:

$$\big(r, \eta'_s(r)\big) = \big(r, \eta'(r)\big)\big(e, d(s^{-1})\big) = \big(r, \alpha_r\big(\alpha_{r^{-1}}(\eta'(r)d(s^{-1}))\big)\big).$$

Then $\gamma_s\eta_s(r) = \big(r, \gamma'_s\eta'_s(r)\big) = \big(r, \Delta(s)^{1/2}\alpha_{rs}\big(\alpha_{s^{-1}r^{-1}}(\eta'(rs)d(s^{-1}))\big)\big)$. Hence the map $G\times G \to A$ such that $(s,r) \longmapsto \gamma'_s\eta'_s(r)$ is continuous and has compact support. Therefore the map $G \to C_0(A)$ given by $s \longmapsto \gamma'_s\eta'_s$ is continuous. It follows that $\|\gamma_s\eta_s - \gamma_t\eta_t\|_\infty \to 0$ when $s \to t$. So we also have that $\|\gamma_s\eta_s - \gamma_t\eta_t\| \to 0$ if $s \to t$. Hence there exists i_2 such that $\forall i \geq i_2$:

$$\|\gamma_{t_i}\eta_{t_i} - \gamma_t\eta_t\| \leq \epsilon/4 \tag{5}$$

On the other hand, since $\xi_i \to \xi$, there exists i_3 such that $\forall i \geq i_3$ we have:

$$\|\xi_i - \xi\| < \epsilon/4 \tag{6}$$

Let $i_0 \geq i_1, i_2, i_3$. Since each γ_s is an isometry, if $i \geq i_0$, by (3), (4), (5) and (6) we have that:

$$\begin{aligned}\|\gamma_{t_i}(\xi_i) - \gamma_t(\xi)\| &\leq \|\gamma_{t_i}(\xi_i) - \gamma_{t_i}(\eta_{t_i})\| + \|\gamma_{t_i}(\eta_{t_i}) - \gamma_t(\eta_t)\| + \|\gamma_t(\eta_t) - \gamma_t(\xi)\|\\ &\leq \|\xi_i - \eta_{t_i}\| + \epsilon/4 + \|\eta a - \xi\|\\ &\leq \|\xi_i - \xi\| + \|\xi - \eta_{t_i}\| + \epsilon/2\\ &< \epsilon\end{aligned}$$

This proves that γ is continuous, which ends the proof. □

Our next task is to show that, up to Morita equivalence, the Morita enveloping action is unique. This will be proved in Proposition 7.6.

Theorem 7.4. *Let $\mathcal{B} = (B_t)_{t\in G}$ be a Fell bundle over G, $\mathcal{E} = (E_t)_{t\in G}$ a right ideal of $\mathcal{B}$, and $\mathcal{A} = (A_t)_{t\in G}$ a sub–Fell bundle of $\mathcal{B}$ contained in $\mathcal{E}$. Define $\Bbbk_c(\mathcal{E}) = \{k \in \Bbbk_c(\mathcal{B}) :\ k(r,s) \in E_{rs^{-1}}, \forall r, s \in G\}$, and let $\Bbbk(\mathcal{E})$ be the closure of $\Bbbk_c(\mathcal{E})$ in $\Bbbk(\mathcal{B})$. Suppose that $\mathcal{A}\mathcal{E} \subseteq \mathcal{E}$ and $\mathcal{E}\mathcal{E}^* \subseteq \mathcal{A}$. Then:*

1. $\Bbbk_c(\mathcal{A})\Bbbk_c(\mathcal{E}) \subseteq \Bbbk_c(\mathcal{E})$, *and* $\Bbbk(\mathcal{A})\Bbbk(\mathcal{E}) \subseteq \Bbbk(\mathcal{E})$
2. $\Bbbk_c(\mathcal{E})\Bbbk_c(\mathcal{B}) \subseteq \Bbbk_c(\mathcal{E})$, *and* $\Bbbk(\mathcal{E})\Bbbk(\mathcal{B}) \subseteq \Bbbk(\mathcal{E})$.
3. $\Bbbk_c(\mathcal{E})\Bbbk_c(\mathcal{E})^* \subseteq \Bbbk_c(\mathcal{A})$, *and* $\Bbbk(\mathcal{E})\Bbbk(\mathcal{E})^* = \Bbbk(\mathcal{A})$. *In particular,* $\Bbbk(\mathcal{A})$ *is a hereditary sub-*C^**-algebra of* $\Bbbk(\mathcal{B})$.
4. *If* $\operatorname{span}(\mathcal{E}^*\mathcal{E} \bigcap B_t)$ *is dense in* B_t*, for every* $t \in G$*, then the closure of* $\operatorname{span}\Bbbk_c(\mathcal{E})^*\Bbbk_c(\mathcal{E})$ *in* $\Bbbk_c(\mathcal{B})$ *in the inductive limit topology is* $\Bbbk_c(\mathcal{B})$*, and* $\overline{\operatorname{span}}\,\Bbbk(\mathcal{E})^*\Bbbk(\mathcal{E}) = \Bbbk(\mathcal{B})$. *In particular,* $\Bbbk(\mathcal{A})$ *and* $\Bbbk(\mathcal{B})$ *are Morita equivalent.*

Moreover, $\Bbbk(\mathcal{E})$ *is invariant under the natural action of* G *on* $\Bbbk(\mathcal{B})$. *In the hipotheses of* 4. *above, the natural actions of* G *on* $\Bbbk(\mathcal{A})$ *and on* $\Bbbk(\mathcal{B})$ *are Morita equivalent.*

Proof. As we have already remarked at the end of Section 6, if $\mathcal{A} \subseteq \mathcal{B}$, then $\Bbbk(\mathcal{A}) \subseteq \Bbbk(\mathcal{B})$.

Let $a \in \Bbbk_c(\mathcal{A})$, ξ, $\eta \in \Bbbk_c(\mathcal{E})$, $b \in \Bbbk_c(\mathcal{B})$, r, $s \in G$. Then:
1. $a * \xi(r,s) = \int_G a(r,t)\xi(t,s)dt \in A_{rt^{-1}}E_{ts^{-1}} \subseteq \mathcal{E} \bigcap B_{rs^{-1}} = E_{rs^{-1}}$. Therefore, $\Bbbk_c(\mathcal{A})\Bbbk_c(\mathcal{E}) \subseteq \Bbbk_c(\mathcal{E})$, and hence $\Bbbk(\mathcal{A})\Bbbk(\mathcal{E}) \subseteq \Bbbk(\mathcal{E})$.
2. Note that the second assertion is a consequence of the first one. On the other hand: $\xi * b(r,s) = \int_G \xi(r,t)b(t,s)dt \in E_{rt^{-1}}B_{ts^{-1}} \subseteq \mathcal{E} \bigcap B_{rs^{-1}} = E_{rs^{-1}}$, which proves the first assertion of 2.
3. $\xi * \eta^*(r,s) = \int_G \xi(r,t)\eta(s,t)^*dt \in E_{rt^{-1}}E_{ts^{-1}} \subseteq \mathcal{A} \bigcap B_{rs^{-1}}$. Thus, $\Bbbk_c(\mathcal{E})\Bbbk_c(\mathcal{E})^* \subseteq \Bbbk_c(\mathcal{A})$ e $\Bbbk(\mathcal{E})\Bbbk(\mathcal{E})^* \subseteq \Bbbk(\mathcal{A})$. The equality follows from the Cohen–Hewitt theorem and from the inclusion $\Bbbk(\mathcal{A})\Bbbk(\mathcal{A})^* \subseteq \Bbbk(\mathcal{E})\Bbbk(\mathcal{E})^*$.
4. Let $F = \operatorname{span}\Bbbk_c(\mathcal{E})^*\Bbbk_c(\mathcal{E}) \subseteq \Bbbk_c(\mathcal{B})$, and consider the dense subalgebra $\Theta = C_c(G) \bigodot C_c(G)$ of $C_c(G \times G)$. If φ, $\psi \in C_c(G)$, ξ, $\eta \in \Bbbk_c(\mathcal{E})$, then:

$$(\varphi \otimes \psi)\xi^* * \eta(r,s) = \int_G \varphi(r)\psi(s)\xi(t,r)^*\eta(t,s)dt = \int_G \big((\bar{\varphi}\xi)(t,r)\big)^*(\psi\eta)(t,s)dt = (\bar{\varphi}\xi)^* * (\psi\eta)(r,s),$$

where $\varphi\xi(r,s) = \varphi(r)\xi(r,s)$. Since $\varphi\xi \in \Bbbk_c(\mathcal{E})$, $\forall \varphi \in C_c(G)$, $\xi \in \Bbbk_c(\mathcal{E})$, we see that $\Theta F \subseteq F$.

Now, let $a \in B_{rs^{-1}}$ be such that $a = b^*c$, with $b \in E_{tr^{-1}}$ and $c \in E_{ts^{-1}}$, for some $t \in G$. Let ξ, $\eta \in \Bbbk_c(\mathcal{E}) = C_c(\mathcal{E}_\nu)$ be such that $\xi(t,r) = b$, $\eta(t,s) = c$, so $\xi(t,r)^*\eta(t,s) = b^*c = a$. Let $\mathcal{V}$ be a base of neighborhoods of the identity of G, and $(f_V)_{V\in\mathcal{V}}$ an approximate identity of $L^1(G)$ as in 4.14. For each $V \in \mathcal{V}$ consider $\eta_V^t \in \Bbbk_c(\mathcal{E})$ such that $\eta_V^t(r,s) = f_V(t^{-1}r)\eta(r,s)$. The map $G \to B_{rs^{-1}}$ given by $u \longmapsto \xi(u,r)^*\eta(u,s)$ is continuous with compact support. Thus, if $\epsilon > 0$, there exists a neighborhood V_ϵ of the identity of G such that, if $t^{-1}u \in V_\epsilon$, then $\|\xi(u,r)^*\eta(u,s) - \xi(t,r)^*\eta(t,s)\| < \epsilon$. Hence, if $V \subseteq V_\epsilon$:

$$\begin{aligned}
\|\xi^* * \eta_V^t(r,s) - a\| &= \|\int_G f_V(t^{-1}u)[\xi(u,r)^*\eta(u,s) - \xi(t,r)^*\eta(t,s)]du\| \\
&\leq \int_{tV} f_V(t^{-1}u)\|\xi(u,r)^*\eta(u,s) - \xi(t,r)^*\eta(t,s)\|du \\
&< \int_G f_V(t^{-1}u)\epsilon du \\
&= \epsilon
\end{aligned}$$

Since $B_{rs^{-1}} = \overline{\operatorname{span}}\{E_{ur^{-1}}^*E_{us^{-1}} : u \in G\}$, we conclude that $\operatorname{span}\Bbbk_c(\mathcal{E})^*\Bbbk_c(\mathcal{E}) \bigcap B_{rs^{-1}}$ is dense in $B_{rs^{-1}}$, $\forall r, s \in G$, and hence that $\operatorname{span}\Bbbk_c(\mathcal{E})^*\Bbbk_c(\mathcal{E})$ is dense in $\Bbbk_c(\mathcal{B})$ in the inductive limit topology by 6.4. This, together with 3., proves that $\Bbbk(\mathcal{A})$ and $\Bbbk(\mathcal{B})$ are Morita equivalent.

Let β be the natural action of G on $\Bbbk(\mathcal{B})$, and $k \in \Bbbk_c(\mathcal{E})$, $t \in G$. Then $\beta_t(k)\big|_{(r,s)} = \Delta(t)k(rt,st) \in E_{rt(st)^{-1}} = E_{sr^{-1}}$, and therefore $k \in \Bbbk_c(\mathcal{E})$. It follows that $\beta_t(\Bbbk_c(\mathcal{E})) = \Bbbk_c(\mathcal{E})$, thus $\beta_t(\Bbbk(\mathcal{E})) = \Bbbk(\mathcal{E})$. This completes the proof. ∎

Corollary 7.5. *Let* α *and* β *be Morita equivalent partial actions with associated Fell bundles* $\mathcal{B}_\alpha$ *and* $\mathcal{B}_\beta$ *respectively. Let* $\check{\alpha}$ *and* $\check{\beta}$ *be the natural actions on* $\Bbbk(\mathcal{B}_\alpha)$ *and* $\Bbbk(\mathcal{B}_\beta)$. *Then* $\check{\alpha}$ *and* $\check{\beta}$ *are Morita equivalent. In particular,* $\Bbbk(\mathcal{B}_\alpha)$ *and* $\Bbbk(\mathcal{B}_\beta)$ *are Morita equivalent.*

Proof. Suppose that γ is a partial action that implements the equivalence between α and β. Since the Morita equivalence of partial actions is transitive, we may replace β by the linking partial action of γ (see the proof of 5.15 in page 19). In the proof of Proposition 5.15, we constructed a right ideal $\mathcal{E}$ of $\mathcal{B}_\beta$ such that $\mathcal{B}_\alpha$, $\mathcal{E}$ and $\mathcal{B}_\beta$ satisfy the conditions 1.–4. of Theorem 7.4 (with $\mathcal{B}_\alpha$ in place of $\mathcal{A}$ and $\mathcal{B}_\beta$ instead of $\mathcal{B}$). Now the result follows from 7.4. $\square$

Proposition 7.6. *The Morita enveloping action is unique, up to Morita equivalence.*

Proof. Since the Morita equivalence of partial actions is transitive, by Corollary 7.5 above it is enough to show that if (β, B) is an enveloping action of the partial action (α, A) and $\breve{\alpha}$ is the natural action on $\Bbbk(\mathcal{A})$, where $\mathcal{A}$ is the associated Fell bundle of α, then β and $\breve{\alpha}$ are Morita equivalent.

Now, let $\mathcal{B}$ be the Fell bundle associated with the enveloping action β, and $\breve{\beta}$ the natural action of G on $\Bbbk(\mathcal{B})$. Consider the right ideal $\mathcal{E}$ of $\mathcal{B}$ constructed in the proof of Theorem 4.18, that is, $\mathcal{E} = \{(t,x) \in \mathcal{B} : \ x \in A, \forall t \in G\}$. Applying Theorem 7.4, it follows that $\breve{\alpha}$ and $\breve{\beta}$ are Morita equivalent. Since $\mathcal{B}$ is a saturated Fell bundle. by 6.6 we have that $\Bbbk(\mathcal{B}) = I$, and therefore $\breve{\beta} \overset{M}{\sim} \beta$ by 7.3 2. and 3. In conclusion: $\breve{\alpha} \overset{M}{\sim} \beta$. $\square$

Corollary 7.7. *Let (α, A) be a partial action with Morita enveloping action (β, B), and assume that $\mathcal{P}$ is an ideal property. Then A has property $\mathcal{P}$ if and only if B has property $\mathcal{P}$. In particular, B is nuclear, liminal, antiliminal or postliminal, if and only if A is respectively nuclear, liminal, antiliminal or postliminal.*

Proof. This is a direct consequence of 6.17, 7.3 and 7.6. $\square$

8. Partial actions induced on $\hat{A}$ and $\mathrm{Prim}(A)$

Let α be a partial action on the C^*-algebra A that has a Morita enveloping action β acting on a C^*-algebra B. One can see that α induces partial actions on $\hat{A}$ and $\mathrm{Prim}(A)$, the spectrum of A and the primitive ideal space of A respectively. In this part we will show that the enveloping actions of these partial actions are precisely the actions induced by β on $\hat{B}$ and $\mathrm{Prim}(B)$ respectively.

Let us fix some notation. If $I \lhd A$ let $\mathcal{O}_I = \{P \lhd A : \ P \text{ is primitive and } P \not\supseteq I\}$, that is, $\{\mathcal{O}_I : \ I \lhd A\}$ is the Jacobson topology of $\mathrm{Prim}(A)$. Recall that there is a natural map $\kappa : \hat{A} \to \mathrm{Prim}(A)$, given by $\kappa([\pi]) = \ker \pi$, that is surjective but in general not injective. The topology of $\hat{A}$ is the initial topology defined by κ, so the open sets are of the form $\mathcal{V}_I = \kappa^{-1}(\mathcal{O}_I)$. With this topology $\hat{A}$ is locally compact, and it is compact if A is unital ([14], VII-6.11). The restriction maps $r : \mathcal{O}_I \to \mathrm{Prim}(I)$, such that $r(P) = P \cap I$, and $r : \mathcal{V}_I \to \hat{I}$, such that $r([\pi]) = [\pi\big|_I]$, are homeomorphisms.

Proposition 8.1. *Let $\beta : G \times B \to B$ be a continuous action of G on the C^*-algebra B. Then:*

1. *$\hat{\beta} : G \times \hat{B} \to \hat{B}$ such that $\hat{\beta}_t([\pi]) = [\pi \circ \beta_{t^{-1}}]$ is a continuous action.*
2. *$\tilde{\beta} : G \times \mathrm{Prim}(B) \to \mathrm{Prim}(B)$ such that $\tilde{\beta}_t(P) = \beta_t(P)$ is a continuous action.*

Proof. A proof of 1. may be found in [26], 7.1. To prove 2., note that the following diagram commutes:

$$\begin{tikzcd} G \times \hat{B} \arrow[r, "id\times\kappa"] \arrow[d, "\hat{\beta}"'] & G \times \mathrm{Prim}(B) \arrow[d, "\tilde{\beta}"] \\ \hat{A} \arrow[r, "\kappa"'] & \mathrm{Prim}(B) \end{tikzcd}$$

Since $\hat{B}$ has the initial topology induced by κ, then $\mathrm{Prim}(B)$ is precisely the topological quotient space of $\hat{B}$ with respect to κ, and therefore $G \times \mathrm{Prim}(B)$ is the topological quotient space of $G \times \hat{B}$ with respect to $id \times \kappa$. Thus, $\tilde{\beta}$ is continuous if and only if $\tilde{\beta}(id \times \kappa)$ is continuous; but $\tilde{\beta}(id \times \kappa) = \kappa\hat{\beta}$, and $\kappa\hat{\beta}$ is continuous. $\square$

It is possible to give a direct proof of the following result, but we will obtain it indirectly from 8.3 and 8.5.

Proposition 8.2. *Let $\alpha = (\{D_t\}_{t\in G}, \alpha_{tt\in G})$ be a partial action G on the C^*-algebra A.*

1. *For each $t \in G$ let $\mathcal{O}_t := \mathcal{O}_{D_t} = \{P \in \mathrm{Prim}(A) : P \not\supseteq D_t\}$. Then $\tilde{\alpha} = (\{\mathcal{O}_t\}_{t\in G}, \{\tilde{\alpha}_t\}_{t\in G})$ is a set theoretic partial action of G on $\mathrm{Prim}(A)$, where $\tilde{\alpha}_t(P)$ is the unique primitive ideal of A such that $\tilde{\alpha}_t(P) \cap D_t = \alpha_t(P \cap D_{t^{-1}})$.*
2. *For each $t \in G$, let $\mathcal{V}_t := \{[\pi] \in \hat{A} : \pi\big|_{D_t} \neq 0\}$, that is, $\mathcal{V}_t = \kappa^{-1}(\mathcal{O}_t)$, where $\mathcal{O}_t$ is like in 1. Then $\hat{\alpha} = (\{\mathcal{V}_t\}_{t\in G}, \{\hat{\alpha}_t\}_{t\in G})$ is a set theoretic partial action of G on $\hat{A}$, where, if $[\pi] \in \mathcal{V}_t$, then $\hat{\alpha}_t([\pi])$ is the class of the unique extension of $\pi \circ \alpha_{t^{-1}} : D_t \to B(H_\pi)$ to all of A.*

Suppose that the partial action (α, A) has an enveloping action (β, B). Since $A \lhd B$, $\hat{A}$ is naturally homeomorphic to an open subset of $\hat{B}$, and therefore the action $\hat{\beta}$ restricted to this open set defines, via this homeomorphism, a partial action of G on $\hat{\alpha}$. We show next that this partial action agrees with $\hat{\alpha}$. Similarly, the restriction of $\tilde{\beta}$ to $\mathrm{Prim}(A)$ agrees with $\tilde{\alpha}$.

Proposition 8.3. *Let α be a partial action on the C^*-algebra A, with enveloping action β acting on the C^*-algebra B. Then $(\tilde{\alpha}, \mathrm{Prim}(A))$ and $(\hat{\alpha}, \hat{A})$ are partial actions, and:*

1. *$(\tilde{\beta}, \mathrm{Prim}(B))$ is the enveloping action of $(\tilde{\alpha}, \mathrm{Prim}(A))$, and*
2. *$(\hat{\beta}, \hat{B})$ is the enveloping action of $(\hat{\alpha}, \hat{A})$.*

Proof. We identify $\mathrm{Prim}(A)$ with $\mathcal{O}_A$ through the homeomorphism r^{-1}. Thus $\tilde{\alpha} = (\{\mathcal{O}_t\}, \{\tilde{\alpha}_t\})$ becomes: $\mathcal{O}_t = \{P \in \mathrm{Prim}(B) : P \not\supseteq D_t\}$, and if $P \in \mathcal{O}_{t^{-1}}$, then $\tilde{\alpha}_t(P) \in \mathcal{O}_t$ is the unique primitive ideal of B such that $\tilde{\alpha}_t(P) \cap D_t = \alpha_t(P \cap D_{t^{-1}})$. Let us see first that $\mathrm{dom}(\tilde{\beta}_t\big|_{\mathcal{O}_A}) = \mathrm{dom}\,\tilde{\alpha}_t$:

$$\mathrm{dom}(\tilde{\beta}_t)\big|_{\mathcal{O}_A} = \{P \in \mathrm{Prim}(B) : P \in \mathcal{O}_A \text{ e } \tilde{\beta}_t(P) \in \mathcal{O}_A\} = \mathcal{O}_A \cap \mathcal{O}_{\tilde{\beta}_{t^{-1}}(A)} = \mathcal{O}_{A \cap \tilde{\beta}_{t^{-1}}(A)} = \mathrm{dom}(\tilde{\alpha}_t).$$

Now, if $P \in \mathcal{O}_{t^{-1}}$, we have that $\tilde{\beta}_t(P) = \beta_t(P) \not\supseteq D_t$, and hence $\tilde{\beta}_t(P) \in \mathcal{O}_t$. But $\beta_t(P) \cap D_t = \beta_t(P \cap \beta_{t^{-1}}(D_t)) = \alpha_t(P \cap D_{t^{-1}})$. Then $\tilde{\beta}_t(P)$ agrees with $\tilde{\alpha}_t(P)$; in particular $\tilde{\alpha}$ is a partial action.

It remains to verify that the $\tilde{\beta}$-orbit of $\mathcal{O}_A$ is $\mathrm{Prim}(B)$. Suppose that there exists $P \in \mathrm{Prim}(B)$ such that $P \neq \tilde{\beta}_t(Q)$, $\forall Q \in \mathcal{O}_A$. Then $\tilde{\beta}_t(P) \notin \mathcal{O}_A$, $\forall t \in G$, that is, $\beta_t(P) \supseteq A$, $\forall t \in G$. But then $P \supseteq \beta_t(A)$, $\forall t \in G$, and therefore $P \supseteq \overline{[\beta(A)]} = B$, because β is the enveloping action of α. The contradiction implies that every primitive ideal belongs to the $\tilde{\beta}$-orbit of some element of $\mathcal{O}_A$.

As for $\hat{\beta}$ and $\hat{\alpha}$, identify $\hat{A}$ with $\mathcal{V}_A$ via r^{-1}. Then $\hat{\alpha} = (\{\mathcal{V}_t\}, \{\hat{\alpha}_t\})$ becomes: $\mathcal{V}_t = \{[\pi] \in \hat{B} : \pi\big|_{D_t} \neq 0\}$, and for $[\pi] \in \mathcal{V}_{t^{-1}}$, $\hat{\alpha}_t([\pi])$ is the class of the unique extension to B of the irreducible representation $\pi \circ \alpha_{t^{-1}}$ of D_t. From the computations above it follows that $\mathrm{dom}(\hat{\beta})\big|_{\mathcal{V}_A} = \mathcal{V}_{t^{-1}} = \mathrm{dom}(\hat{\alpha}_t)$. On the other hand, if $[\pi] \in \mathcal{V}_{t^{-1}}$, then $(\pi \circ \beta_{t^{-1}})$ is an extension to B of the representation $\pi \circ \alpha_{t^{-1}}$, and therefore $\hat{\beta}_t([\pi])$ agrees with $\hat{\alpha}_t([\pi])$. It remains to show that the $\hat{\beta}$-orbit of $\mathcal{V}_A$ is all of $\hat{B}$, and this is similar to which has been done previously for $\tilde{\alpha}$ and $\tilde{\beta}$: if $[\pi] \notin \hat{\beta}_t(\mathcal{V}_A)$, $\forall t \in G$, then $\pi\big|_{\beta_t(A)} = 0$, $\forall t \in G$, and therefore $\pi = 0$, what is a contradiction. ∎

Lemma 8.4. *Let $\gamma = (\{E_t\}_{t\in G}, \{\alpha_t\}_{t\in G})$ be a partial action of G on a positive C^*-tring E, and let $(\beta, B) = (\gamma^l, E^l)$, $(\alpha, A) = (\gamma^r, E^r)$. Consider the set theoretic partial actions $(\tilde{\beta}, \mathrm{Prim}(B))$ and $(\tilde{\alpha}, \mathrm{Prim}(A))$ induced by β and α on $\mathrm{Prim}(B)$ and $\mathrm{Prim}(A)$ respectively (see Proposition 8.2). Then the Rieffel homeomorphism $\mathsf{R} : \mathrm{Prim}(B) \to \mathrm{Prim}(A)$ is an isomorphism of set theoretic partial actions $\mathsf{R} : \tilde{\beta} \to \tilde{\alpha}$. A similar statement stands for the set theoretic partial actions $\hat{\beta}$ and $\hat{\alpha}$ induced by β and α on the corresponding spectra $\hat{B}$ and $\hat{A}$ of B and A.*

Proof. It is well known that if $P \in \mathrm{Prim}(B)$, then $\mathsf{R}(P) \in \mathrm{Prim}(A)$, and also that R is a homeomorphism (see for instance Corollary 3.33 of [26]). Since $\mathsf{R}(E_t^l) = E_t^r$, it follows that $\mathsf{R}(\mathcal{O}_t^B) = \mathcal{O}_t^A$. Let $I \lhd B$. Then $\beta_t\big(E_{t^{-1}}^l \cap I\big)$ is an ideal of E_t^l, and therefore there exists a unique $F \lhd E_t$ such that $F^l =$

$\beta_t\big(E^l_{t^{-1}}\cap I\big)$. We claim that $F=\gamma_t(E_{t^{-1}}\cap IE)$. Indeed, $\langle F,F\rangle_l=\langle\gamma_t(E_{t^{-1}}\cap IE),\gamma_t(E_{t^{-1}}\cap IE)\rangle_l=\beta_t(\langle E_{t^{-1}}\cap IE,E_{t^{-1}}\cap IE\rangle_l)$, so $F^l=\beta_t(E^l_{t^{-1}}\cap I)$. Similarly, if $J\lhd A$, we have that $\alpha_t\big(E^r_{t^{-1}}\cap J\big)=\big(\gamma_t(E_{t^{-1}}\cap EJ)\big)^r$. Now, if $P\in\mathcal{O}^B_{t^{-1}}$, then $\tilde\beta_t(P)=Q$, where $Q\in\mathrm{Prim}(B)$ is the unique primitive ideal such that $Q\cap E^l_t=\beta_t\big(E^l_{t^{-1}}\cap P\big)$. Since R is a lattice isomorphism, it follows that $\mathsf{R}(Q)\cap\mathsf{R}(E^l_t)=\mathsf{R}\big(\beta_t(E^l_{t^{-1}}\cap P\big)$. That is,

$$\mathsf{R}(Q)\cap E^r_t=\big(E_{t^{-1}}\cap PE\big)^r=\alpha_t\big(E^r_{t^{-1}}\cap(PE)^r\big)=\alpha_t\big(E^r_{t^{-1}}\cap\mathsf{R}(P)\big).$$

Thus, since $\mathsf{R}(Q)$ is a primitive ideal, it must agree with $\tilde\alpha_t\big(\mathsf{R}(P)\big)$. It follows that $\mathsf{R}:\tilde\beta\to\tilde\alpha$ is a morphism of set theoretic partial actions. Similarly, its inverse map $\mathsf{R}:\tilde\alpha\to\tilde\beta$ is a morphism, so R is an isomorphism between the set theoretic partial actions $\tilde\beta$ and $\tilde\alpha$.

The proof of the corresponding statement for $\hat\beta$ and $\hat\alpha$ is similar and it is left to the reader. □

Proposition 8.5. *Let (α,A) be a partial action of G on the C^*-algebra A, and let (β,B) be its Morita enveloping action. Then $(\tilde\alpha,\mathrm{Prim}(A))$ and $(\hat\alpha,\hat A)$ are partial actions, and:*

1. $(\tilde\beta,\mathrm{Prim}(B))$ *is the enveloping action of* $(\tilde\alpha,\mathrm{Prim}(A))$*, and*
2. $(\hat\beta,\hat B)$ *is the enveloping action of* $(\hat\alpha,\hat A)$*.*

Proof. Since β is the Morita enveloping action of α, this one is Morita equivalent to $\beta\big|_I$, for some ideal I of B. By 8.3, $\beta\big|_I$ is a partial action on $\mathrm{Prim}(I)$. On the other hand, R is a homeomorphism, and therefore $\tilde\alpha$ is continuous, that is, $\tilde\alpha$ is a partial action on $\mathrm{Prim}(A)$. Since $(\tilde\beta,\mathrm{Prim}(B))$ is the enveloping action of $(\widetilde{\beta|_I},\mathrm{Prim}(I))$ and R is an isomorphism of partial actions between $(\tilde\beta,\mathrm{Prim}(B))$ and $(\tilde\alpha,\mathrm{Prim}(A))$, it follows that $(\tilde\beta,\mathrm{Prim}(B))$ is the enveloping action of $(\tilde\alpha,\mathrm{Prim}(A))$. The proof of 2. is similar. □

Corollary 8.6. *If (α,A) is a partial action with Morita enveloping action (β,B), then:*

$$\hat A=\mathrm{Prim}(A)\iff\hat B=\mathrm{Prim}(B).$$

Proof. By 8.5 we have that $(\hat\beta,\hat B)$ is the enveloping action of $\hat\alpha$ and that $(\tilde\beta,\mathrm{Prim}(B))$ is the enveloping action of $\tilde\alpha$. Suppose that $\hat\alpha=\tilde\alpha$. Since the enveloping action is unique, we have that $(\hat\beta,\hat B)=(\tilde\beta,\mathrm{Prim}(B))$. The converse is clear. □

The following result may be thought of as a non–commutative version of the well known fact that the integral curves of a vector field on a compact manifold X are defined on all of $\mathbb{R}$.

Corollary 8.7. *Suppose that α is a partial action of G on the C^*-algebra A, with Morita enveloping action (β,B). If $\mathrm{Prim}(A)$ is compact (this is true if A is unital), then there exists an open subgroup H of G such that the restriction of α to H is a global action. In particular, if G is a connected group, then α is a global action.*

Proof. If A is unital, then $\mathrm{Prim}(A)$ is compact by [14], VII-6.11. Now, if $\mathrm{Prim}(A)$ is compact, by 2.4 there exists an open subgroup H of G such that $\tilde\alpha$ restricted to H is a global action on $\mathrm{Prim}(A)$, and therefore every primitive ideal of A is in the domain of $\tilde\alpha_s$, $\forall s\in H$. By the definition of $\tilde\alpha$, this implies that there is no primitive ideal of A containing the ideal $D_{s^{-1}}$, and hence $D_{s^{-1}}=A$, $\forall s\in H$. Therefore $\alpha\big|_{H\times A}$ is a global action. If G is connected, then $G=H$. □

9. Takai duality for partial actions

In this last section of the paper we relate our previous results on enveloping actions with Takai duality for partial actions. If we tried to translate naively Takai duality from the context of global actions to our case of partial ones, it should be expressed as follows: if α is a partial action of G on A and δ is the dual coaction of G on $A\rtimes_{\alpha,r}G$, then A and $A\rtimes_{\alpha,r}G\rtimes_\delta\hat G$ are Morita equivalent. However, as pointed

out by Quigg in [23], this version of Takai duality fails for partial actions, and its failure is proportional to the "partialness" of α.

In what follows we show that we still have a Takai duality for partial actions, which may be briefly expressed in this manner: if $\hat{\delta}$ is the dual action of G on $A \rtimes_{\alpha,r} G \rtimes_{\delta} \hat{G}$, then $\big(\hat{\delta}, A \rtimes_{\alpha,r} G \rtimes_{\delta} \hat{G}\big)$ is the Morita enveloping action of (α, A). In other words, A is no longer Morita equivalent to $A \rtimes_{\alpha,r} G \rtimes_{\delta} \hat{G}$, but to an ideal I of this double crossed product, in such a way that this ideal together with the dual action allow us to recover the whole double crossed product.

To be more precise, we will prove that if $\mathcal{B}_\alpha$ is the Fell bundle associated to the partial action α, and $\breve{\alpha}$ is the natural action on $\Bbbk(\mathcal{B}_\alpha)$, then the dynamical systems $\big(\breve{\alpha}, \Bbbk(\mathcal{B}_\alpha)\big)$ and $\big(\hat{\delta}, A \rtimes_{\alpha,r} G \rtimes_{\delta} \hat{G}\big)$ are isomorphic.

Before proceeding, let us recall some basic facts about crossed products by coactions; for more details, the reader is referred to [15] and [17]. The tensor product we consider is the minimal tensor product. If A and B are C^*-algebras, set $\tilde{M}(A \bigotimes B) := \{m \in M(A \bigotimes B) : m(1 \otimes B) + (1 \otimes B)m \subseteq A \bigotimes B\}$. Let $w_G : L^2(G \times G) \to L^2(G \times G)$ be the unitary operator such that $w_G(\xi)\big|_{(r,s)} = \xi(r, r^{-1}s)$. Its adjoint is $w_G^*(\xi)\big|_{(r,s)} = \xi(r, rs)$. The *comultiplication* on $C_r^*(G)$ is $\delta_G : C_r^*(G) \to \tilde{M}\big(C_r^*(G) \bigotimes C_r^*(G)\big)$ given by $\delta_G(x) = w_G(x \otimes 1)w_G^*$, $\forall x \in C_r^*(G)$. A *coaction* of the locally compact group G on a C^*-algebra A is an injective homomorphism $\delta : A \to \tilde{M}\big(A \bigotimes C_r^*(G)\big)$ such that for any approximate unit $(e_i)_{i\in I}$ of A we have that $\delta(e_i)$ converges strictly to 1 in $\tilde{M}\big(A \bigotimes C_r^*(G)\big)$, and such that $(\delta \otimes id)\delta = (id \otimes \delta_G)\delta$. In particular, δ_G is a coaction of G on $C_r^*(G)$.

Suppose that $\delta : A \to \tilde{M}\big(A \bigotimes C_r^*(G)\big)$ is a coaction of G on A. The *reduced crossed product* $A \rtimes_{\delta,r} \hat{G}$ of A by the coaction δ is:

$$A \rtimes_{\delta,r} \hat{G} := \big\{\delta(a)(1 \otimes \varphi) : a \in A, \varphi \in C_0(G)\big\} \subseteq M\big(A \bigotimes \mathcal{K}\big),$$

where $\varphi \in C_0(G)$ acts by multiplication on $L^2(G)$.

If δ is a coaction of G on A, then there exists a canonical action $\hat{\delta}$ of G on $A \rtimes_{\delta,r} \hat{G}$. This action is called the *dual action* of G on $A \rtimes_{\delta,r} \hat{G}$, and it is characterized by the fact that it verifies $\hat{\delta}_t\big(\delta(a)(1 \otimes \varphi)\big) = \delta(a)(1 \otimes \varphi_t)$ where, if $\varphi \in C_0(G)$, then $\varphi_t(s) = \varphi(st)$, $\forall s \in G$.

Now consider a Fell bundle $\mathcal{B} = (B_t)_{t\in G}$, and let $\mathcal{B}_G$ be the trivial Fell bundle over G, that is, the Fell bundle associated with the trivial action of G on $\mathbb{C}$. We have a Fell bundle $\mathcal{B} \bigotimes \mathcal{B}_G$ over $G \times G$, whose reduced C^*-algebra is naturally isomorphic to $C_r^*(\mathcal{B}) \bigotimes C_r^*(G)$. We also have that $L^2(\mathcal{B} \bigotimes \mathcal{B}_G) \cong L^2(\mathcal{B}) \bigotimes L^2(G)$ (see [1] or [2]). Let $w_{\mathcal{B}} : L^2(\mathcal{B} \bigotimes \mathcal{B}_G) \to L^2(\mathcal{B} \bigotimes \mathcal{B}_G)$ be such that $w_{\mathcal{B}}(\xi)\big|_{(r,s)} = \xi(r, r^{-1}s)$. Then $w_{\mathcal{B}}$ is an adjointable operator, in fact unitary, with adjoint $w_{\mathcal{B}}^*$ such that $w_{\mathcal{B}}^*(\xi)\big|_{(r,s)} = \xi(r, rs)$. Now consider the map $\mathcal{L}\big(L^2(\mathcal{B})\big) \to \mathcal{L}\big(L^2(\mathcal{B} \bigotimes \mathcal{B}_G)\big)$ such that $x \longmapsto w_{\mathcal{B}}(x \otimes 1)w_{\mathcal{B}}^*$. Let $\delta_{\mathcal{B}} : C_r^*(\mathcal{B}) \to \tilde{M}\big(C_r^*(\mathcal{B}) \bigotimes C_r^*(G)\big)$ be the restriction of this map to $C_r^*(\mathcal{B})$. Then $\delta_{\mathcal{B}}$ is a coaction of G on $C_r^*(\mathcal{B})$ ([13]), called the *dual coaction* of G on $C_r^*(\mathcal{B})$.

Let $f \in C_c(\mathcal{B})$, $\xi \in C_c(\mathcal{B}) \subseteq L^2(\mathcal{B})$, and $x \in C_c(G) \subseteq L^2(G)$. Then we have:

$$\begin{aligned}
\delta_{\mathcal{B}}(\Lambda_f)(\xi \otimes x)\big|_{(r,s)} &= w_{\mathcal{B}}(\Lambda_f \otimes 1) w_{\mathcal{B}}^*(\xi \otimes x)\big|_{(r,s)} \\
&= (\Lambda_f \otimes 1) w_{\mathcal{B}}^*(\xi \otimes x)\big|_{(r,r^{-1}s)} \\
&= \int_G (\Lambda_{f(t)} \otimes 1) w_{\mathcal{B}}^*(\xi \otimes x)\big|_{(t^{-1}r,r^{-1}s)} dt \\
&= \int_G (\Lambda_{f(t)} \otimes 1)(\xi \otimes x)\big|_{(t^{-1}r,t^{-1}s)} dt \\
&= \int_G f(t)\xi(t^{-1}r) \otimes x(t^{-1}s) dt \\
&= \int_G (\Lambda_{f(t)} \otimes \lambda_t)(\xi \otimes x)\big|_{(r,s)} dt.
\end{aligned}$$

Consider the representation of $\mathcal{B}$ given by $b_t \to \Lambda_{b_t} \otimes \lambda_t \in \mathcal{L}\big(L^2(\mathcal{B}) \bigotimes L^2(G)\big)$. It follows from the computations above that the integrated form of this representation factors through δ. Thus if $\pi : C_r^*(\mathcal{B}) \to B(H)$ is a faithful representation, then $\delta_{\mathcal{B}}(\Lambda_f) = \pi_\lambda$, where π_λ is the integrated representation of $\mathcal{B} \to B\big(L^2(G,H)\big)$ such that $b_t \longmapsto \lambda_t \otimes \pi(b_t)$.

Let α be a partial action, and $\mathcal{B}_\alpha$ its associated Fell bundle. The next result shows the form that Takai duality has for partial actions: $(\beta, \Bbbk(\mathcal{B}))$ and $(\hat{\delta}_{\mathcal{B}}, C_r^*(\mathcal{B}) \rtimes_{\delta_{\mathcal{B}},r} \hat{G})$ are isomorphic dynamical systems.

Proposition 9.1. *Let $\mathcal{B} = (B_t)_{t \in G}$ be a Fell bundle over G, δ the dual coaction of G on $C_r^*(\mathcal{B})$, $\hat{\delta}$ the dual action of G on $C_r^*(\mathcal{B}) \rtimes_{\delta,r} \hat{G}$, and β the natural action of G on $\Bbbk(\mathcal{B})$. Then there exists an isomorphism $\sigma : \Bbbk(\mathcal{B}) \to C_r^*(\mathcal{B}) \rtimes_{\delta,r} \hat{G}$ such that $\sigma\beta_t = \hat{\delta}_t\sigma$, $\forall t \in G$.*

Proof. We may assume, without loss of generality, that $C_r^*(\mathcal{B}) \subseteq B(H)$ non–degenerately, for some Hilbert space H. Therefore also $\mathcal{B} \subseteq B(H)$, and $C_r^*(\mathcal{B}) \rtimes_{\delta,r} \hat{G} \subseteq B\big(L^2(G,H)\big)$.

On the other hand, by 6.7, the inclusion $\mathcal{B} \hookrightarrow B(H)$ defines a faithful representation $\sigma : \Bbbk(\mathcal{B}) \to B\big(L^2(G,H)\big)$ such that, if $k \in \Bbbk_c(\mathcal{B})$, $x \in L^2(G,H)$, then $\sigma(k)\big|_r = \int_G k(r,s)x(s)ds$.

Now, if $f \in C_c(\mathcal{B})$, $\varphi \in C_c(G) \subseteq C_0(G)$, and $\xi \in C_c(G) \subseteq L^2(G)$, $h \in H$, we have:

$$\begin{aligned}
\delta(f)(1 \otimes \varphi)(\xi \otimes h)\big|_r &= \int_G \lambda_s(\varphi\xi)(r) f(s) h ds \\
&= \int_G \varphi(s^{-1}r)\xi(s^{-1}r) f(s) h ds \\
&= \int_G \varphi(t^{-1})\xi(t^{-1}) f(rt) h dt \\
&= \int_G \Delta(s)^{-1}\varphi(s)\xi(s) f(rs^{-1}) h ds \\
&= \sigma(k_{\varphi,f})(\xi \otimes h)\big|_r,
\end{aligned}$$

where $k_{\varphi,f}(r,s) = \Delta(s)^{-1}\varphi(s)f(rs^{-1})$. Thus, $\sigma(k_{\varphi,f}) = \delta(f)(1 \otimes \varphi)$, and hence $\sigma\big(\Bbbk(\mathcal{B})\big) \supseteq C_r^*(\mathcal{B}) \rtimes_{\delta,r} \hat{G}$. To prove the converse inclusion, it is enough to show that $F := \operatorname{span}\{k_{\varphi,f} : \varphi \in C_c(G),\ f \in C_c(\mathcal{B})\}$ is dense in $\Bbbk_c(\mathcal{B})$ in the inductive limit topology. This will follow from Lemma 6.4. For given $\phi, \psi \in C_c(G)$, let $\phi \diamond \psi : G \times G \to \mathbb{C}$ be such that $\phi \diamond \psi(r,s) = \phi(s)\psi(rs^{-1})$. It is clear that $\phi \diamond \psi$ is continuous and has compact support. Let $\Theta := \operatorname{span}\{\psi_1 \diamond \psi_2 : \psi_1, \psi_2 \in C_c(G)\}$. Then Θ is dense in $C_0(G \times G)$ by the Stone–Weierstrass theorem. In particular, Θ is a dense subspace $C_c(G \times G)$ in the inductive limit topology. Let us see that $\Theta F \subseteq F$: if $\psi_1 \diamond \psi_2 \in \Theta$, $k_{\varphi,f} \in F$, we have:

$$(\psi_1 \diamond \psi_2) k_{\varphi,f}(r,s) = \Delta(s)^{-1}\psi_1(s)\psi_2(rs^{-1})\varphi(s)f(rs^{-1}) = k_{\psi_1\varphi,\psi_2 f}(r,s).$$

Now, it is clear that $F(r,s) = B_{rs^{-1}}$, $\forall r,s \in G$. This shows that F is dense in $\Bbbk(\mathcal{B})$, and therefore $\sigma : \Bbbk(\mathcal{B}) \to C_r^*(\mathcal{B}) \rtimes_{\delta,r} \hat{G}$ is an isomorphism. Finally, if $t \in G$ we have:

$$\sigma^{-1}\big(\hat{\delta}_t(\delta(f)(1 \otimes \varphi)\big) = k_{\varphi_t,f}.$$

On the other hand:

$$\beta_t(k_{\varphi,f})\big|_{(r,s)} = \Delta(t)k_{\varphi,f}(rt,st) = \Delta(t)\Delta(st)^{-1}\varphi(st)f(rs^{-1}) = \Delta(s)^{-1}\varphi_t(s)f(rs^{-1}) = k_{\varphi_t,f}\big|_{(r,s)},$$

so $\sigma\beta_t(k_{\varphi,f}) = \hat{\delta}_t\sigma(k_{\varphi,f})$, and since F is dense in $\Bbbk(\mathcal{B})$, it follows that $\sigma\beta_t = \hat{\delta}_t\sigma$. □

Remark 9.2. Since $\Bbbk(\mathcal{B})$ is isomorphic to $C_r^*(\mathcal{B})\rtimes_{\delta,r}\hat{G}$, we may apply Corollary 7.7 (or 6.17) to $C_r^*(\mathcal{B})\rtimes_{\delta,r} \hat{G}$. In particular, we have that $C_r^*(\mathcal{B}) \rtimes_{\delta,r} \hat{G}$ is nuclear, liminal, antiliminal or postliminal, if and only if B_e is respectively nuclear, liminal, antiliminal or postliminal. This was already known for discrete groups ([22],[20]).

We also conclude that if $\mathcal{A}$ and $\mathcal{B}$ satisfy conditions 1.–3. in Theorem 7.4, then $C_r^*(\mathcal{A}) \rtimes_{\delta,r} \hat{G}$ is an hereditary sub-C^*-algebra of $C_r^*(\mathcal{B})\rtimes_{\delta,r}\hat{G}$. If they also satisfy condition 4. in 7.4, then $\big(C_r^*(\mathcal{A})\rtimes_{\delta,r}\hat{G},\hat{\delta}\big)$ and $\big(C_r^*(\mathcal{B}) \rtimes_{\delta,r} \hat{G},\hat{\delta}\big)$ are Morita equivalent dynamical systems. In particular, if (α,A) and (β,B) are Morita equivalent partial actions, then $\big(A\rtimes_{\alpha,r}G\rtimes_{\delta,r}\hat{G},\hat{\delta}\big)$ and $\big(B\rtimes_{\beta,r}G\rtimes_{\delta,r}\hat{G},\hat{\delta}\big)$ are Morita equivalent.

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